STO6 LECTURE 15

Note Title 4/6/2006

Today

- Feedback Capacity
- Joint Source - Channel Coding
- Continuous Channels ...

Review of last lecture

in binary symmetric channel with crossover probability p ....

$$\text{Capacity} = C = 1 - H(p)$$

if we transmit at rate $R < C$

then err w.p. $P_{err} = 2^{-E_{RCE}(R) \cdot n}$

where

$$E_{RCE}(R) = R_0 - R \quad \text{if } R < R_{crit}$$

$$= D(p_R \| p) \quad \text{if } R > R_{crit}$$

$R_{crit} = \ldots$

$R_0 = \ldots$

$p_R = H^{-1}(1 - R)$

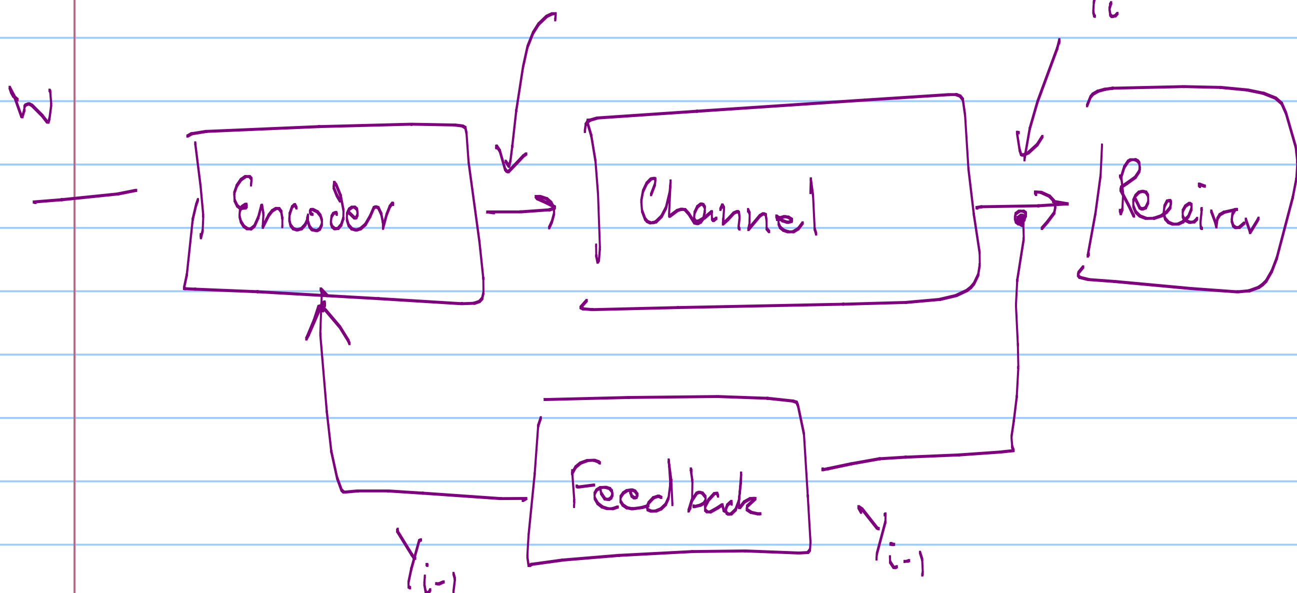
Punchline error is exponentially small.

Today: Couple of conclusions

- What if channel allows feedback?
- Joint source-channel encoding.

FEEDBACK

$$X_i = E(W, Y^{i-1})$$



Suppose channel has feedback, but is otherwise unchanged.

Does capacity change?

Let C = capacity w.o. feedback

C_{FB} = capacity w. feedback

Obvious: $C_{FB} \geq C$ [Encoder ignores feedback]

Not Obvious: $C_{FB} \leq C$

Proof: goes back to definition of C.

$$nR = H(W|Y^n) + I(W;Y^n)$$

$$H(W|Y^n) \leq 1 + P_e^{(n)} \cdot n \cdot R \quad \text{[Fano]}$$

$$\Rightarrow nR \leq \underbrace{1 + \cdots} + \underset{\substack{\uparrow \\ \text{essentially capacity.}}}{I(W;Y^n)}$$

$$I(W;Y^n) = H(Y^n) - H(Y^n|W)$$

$$= H(Y^n) - \sum_i H(Y_i | Y_1 \ldots Y_{i-1}, W)$$

$$= H(Y^n) - \sum_i H(Y_i | Y_1 \cdots Y_{i-1}, W, X_i)$$

$$= H(Y^n) - \sum_i H(Y_i | X_i)$$

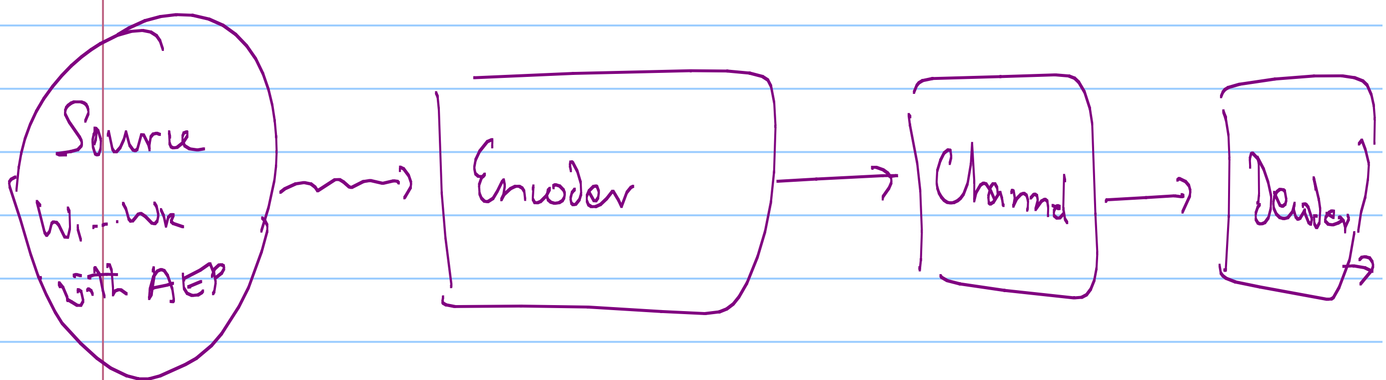
$$H(Y^n) \leq \sum_i H(Y_i)$$

$$\Rightarrow I(W;Y^n) \leq \sum_{i=1}^{n} I(Y_i ; X_i)$$

$$\leq n \cdot C$$

$$\Rightarrow C_{FB} \leq C .$$

Joint-Source Channel Coding



When is transmission possible?

Suppose entropy rate of source $= H(\mathcal{W})$

Capacity of Channel $= C$

& then ~~coding~~ comm. possible iff $H(\mathcal{W}) < C$.

Proof: Exercise.

New Topic: Continuous Channels

"Telephone Wire"

Can feel any voltage between $[-1, +1]$

Receive voltage $\in \mathbb{R}$.

What is the capacity?

Idea 1: Transmit uniform real number in the range $\left\{-\frac{\ell}{\ell}, \frac{-\ell+1}{\ell}, \ldots \frac{-1}{\ell}, 0, \frac{1}{\ell}, \ldots \frac{\ell}{\ell}\right\}$

Reception perfect $\Rightarrow$ transmit $\sim \log_2 \ell$ bits
per step.

But we can choose any ℓ we want!

$\Rightarrow$ letting $\ell \to \infty$ capacity $= \infty$.

is this right?

Well ... have to cope with error ....

Error = ?

Say, Each time step uniform additive noise
in $[-\epsilon, +\epsilon]$

(transmit $x \to$ receive $x+\delta$, $\delta \in_U [-\epsilon, \epsilon]$)

Can still use channel . . . .

- transmit uniform dist of $\underbrace{\{-1, -1+2\epsilon, -1+4\epsilon \ldots 1\}}_{\frac{1}{\epsilon} \text{ possibilities}}$

- Capacity $\geq \log_2 \frac{1}{\epsilon}$.

Is this right?

Best possible? Should it be finite?

Moral: Need to study continuous channels & continuous errors.

Path: Continuous r.v., (differential) entropy; M.I.; AEP; etc.

Continuous (really mean real-valued) r.v.

X taking values in $\mathbb{R}$

Described by CDF $F_X(x) = P_r[X \le x]$

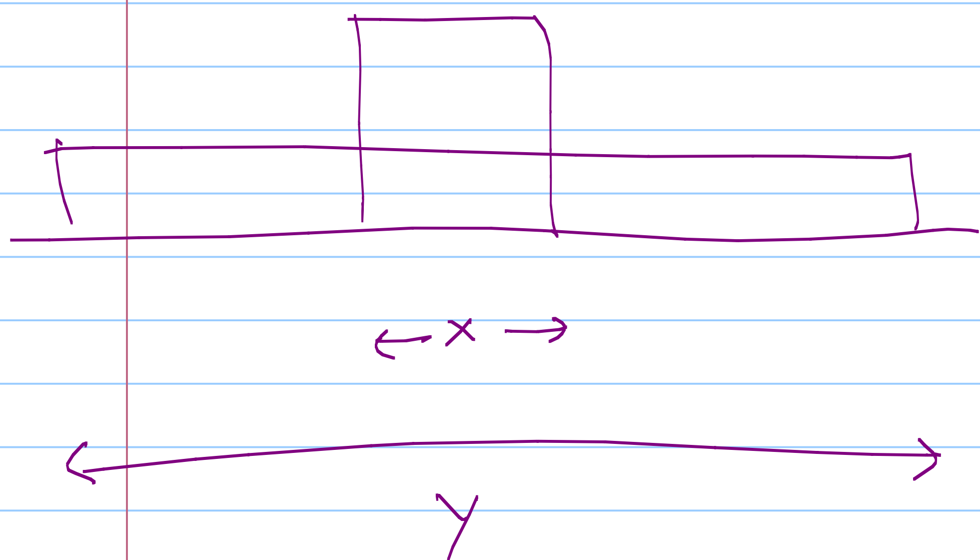
or by pdf $f_X(x) = \frac{d F_X(x)}{dx}$.

<u>Differential Entropy</u>: Measure of uncertainty of continuous random variables ...

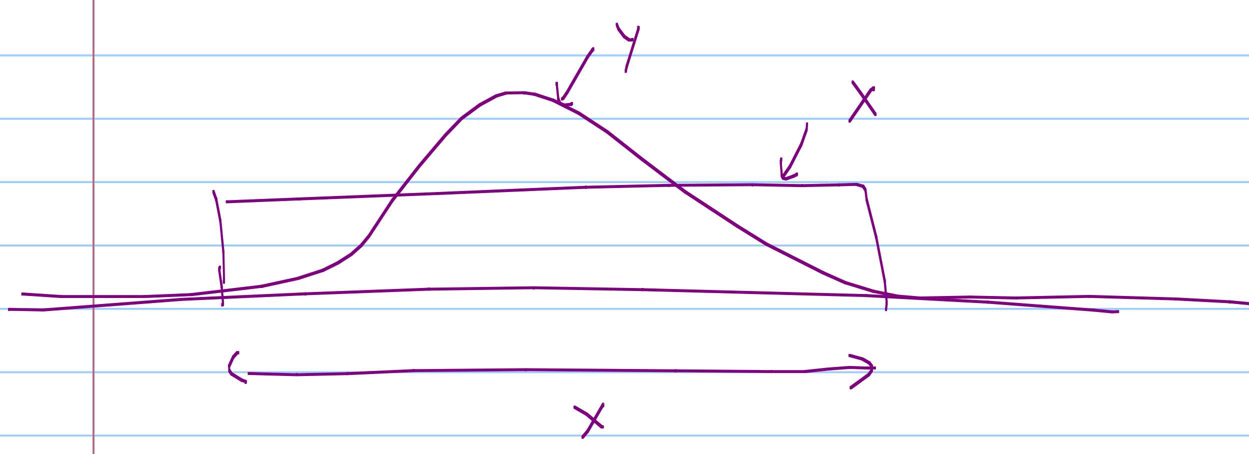
- But continuous r.v. is totally uncertain!
- Does it make sense to ask this question? ...

- Still, want to compare.

Example



Which is more "random"?



Some insight

Convert X, Y into discrete variables

X_ϵ, Y_ϵ by partitioning real

line into intervals of width ϵ

Now can talk about $H(X_\epsilon)$

$H(Y_\epsilon)$

What does

$$\lim_{\epsilon \to 0} \left\{ H(X_\epsilon) - H(Y_\epsilon) \right\}$$ [illegible]?

On one hand $\lim \{ \ \} > 0$

$\Rightarrow$ X more random than Y.

On other hand beyond certain point, for nice X,

$$H(X_{\frac{\epsilon}{2^{\ell}}}) = H(X_\epsilon) + \ell$$

Motivates

$$\lim_{\epsilon \to 0} \left\{ H(X_\epsilon) + \log \epsilon \right\} \stackrel{\Delta}{=} H(x)$$

Turns out $H(x) = -\int_{-\infty}^{\infty} f(x) \log f(x) \, dx$

Example:

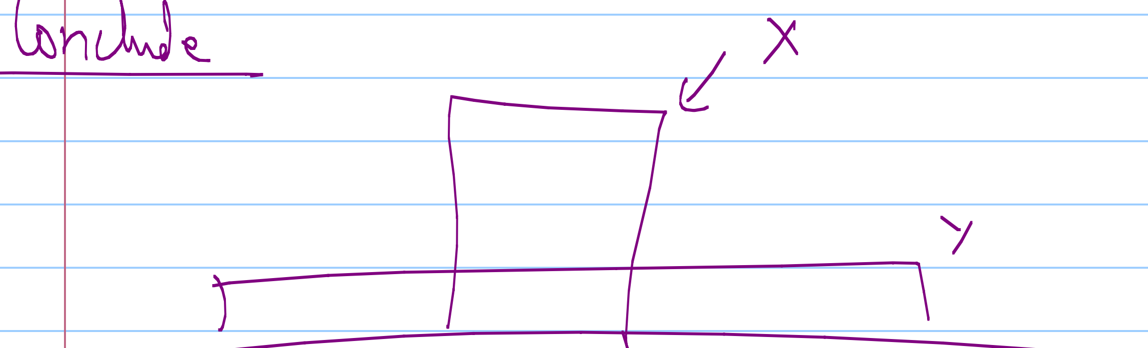
$$X \sim U(a, b)$$

$$f_X(x) = 0 \qquad x < a \text{ or } x > b$$

$$= \frac{1}{b-a} \qquad a \leq x \leq b$$

$$H(X) = \log(b-a)$$

Conclude



Y more random than X.

Gaussian random variable $X \sim N(0, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$h(X) = \int f(x) \cdot \left[\log \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2} \right] dx$$

$$= \log \frac{1}{\sqrt{2\pi\sigma^2}} + \frac{1}{2}$$

$$= \frac{1}{2} \log 2\pi e \sigma^2$$

$Y = X + a \Rightarrow h(Y) = h(X)$

$Y = a \cdot X \Rightarrow h(Y) = h(X) + \log a$

Next lecture

- Joint, Conditional Entropy
- Mutual Information
- Properties

Eventually

Capacity of continuous channels.

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