Lecture 21

Lecturer: Madhu Sudan *Scribe: Vishal Doshi*

1 Review

- Recall the feasible rate region of the multiple access channel (two sources, one receiver) is the convex hull of rates (R_1, R_2) such that $\exists$ distributions p_{X_1} and p_{X_2} with

$$\begin{aligned} R_1 &\le I(X_1; Y|X_2) \\ R_2 &\le I(X_2; Y|X_1) \\ R_1 + R_2 &\le I(X_1, X_2; Y) \end{aligned}$$

- Recall for encoding of two correlated sources, the rate region is given by

$$\begin{aligned} R_1 &\le H(X_1|X_2) \\ R_2 &\le H(X_2|X_1) \\ R_1 + R_2 &\le H(X_1, X_2) \end{aligned}$$

- Thus, any (R_1, R_2) satisfying both sets of equations will be achievable in the general communications problem where we first source code and then channel code. However, this is not the best possible.

  Example. Consider a correlated source with $W_1 = W_2$ distributed as $Bern(\lambda)$ transmitted over an AWGN channel, where the noise has variance σ^2, and the power of each source is constrained to be $\le P$. If we code X_1 and X_2 independently as channel coding requires, we would get a sum rate capacity of $R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2}\right)$. However, if we allow dependency in the channel coding, and let $X_1 = X_2$, then we can get sum rate $R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{4P}{\sigma^2}\right)$.

2 Broadcast Channel

- In the broadcast channel, we have a single center and multiple receivers with various rate requirements. The question we want to ask is what rate requirements are allowable.

- Formally, each subset (indexed by i) of receivers (except the null subset) is interested in a set of messages $S_i = \{1, \ldots, 2^{nR_i}\}$. The encoder takes a message tuple $(w_1, \ldots, w_{2^n-1}) \in S_1 \times \cdots \times S_{2^n-1}$ and produces some x to send over the channel. The channel is defined by a distribution $p_{Y_1,\ldots,Y_n|X}$. Thus the rates $(R_1, \ldots, R_n)$ are achievable if there exists an encoder and n decoders such that the probability of each receiver accurately receiving its relevant information is high.

2.1 Degraded Broadcast Channel

A degraded broadcast channel is one in which $X \to Y_1 \to Y_2$ holds. Note that not every channel has an equivalent degraded broadcast channel.

We think of a BSC degraded broadcast channel as a cascade of two BSC channels with the first output at the output of the first BSC channel and the second as the end of the cascade. Thus we can think of the output at the first as a BSC(p_1) and the second output as a BSC($p_1 * p_2$), where $p_1 * p_2 = p_1(1-p_2) + p_2(1-p_1)$.

For this channel with no joint rate requirements and a particular block length n, we can think of a good coding scheme in which we spread 2^{nR_1} messages so that the Hamming distance between any two of the messages allows us to satisfy our error requirement. Then for each message, we have associated a ball that will decode (for receiver 1) to a given message. In each of these balls, we place smaller balls that will contain the information for receiver 2.

In this setup, for reliable communication to be possible, volume arguments show that if we send at $R_2 = 1 - H(p_1 * p_2)$, we need $2^{nR_1} \leq \frac{2^{nH(p_1 * p_2)}}{2^{nH(p_1)}}$ or $R_1 \leq H(p_1 * p_2) - H(p_1)$.

The proof of the following general result is omitted.

Theorem 1 (Capacity Theorem for Degraded Broadcast Channels) (R_1, R_2) *achievable iff* $\exists U$ *such that* $U \to X \to Y_1 \to Y_2$ *and*

$$R_1 \leq I(X, Y_1 | U)$$
$$R_2 \leq I(U, Y_2)$$