1 Course Overview

- Algebra is the study of sets with binary operations, such as:

<table>
<thead>
<tr>
<th>Set</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>integers</td>
<td>addition &amp; multiplication</td>
</tr>
<tr>
<td>reals</td>
<td>“</td>
</tr>
<tr>
<td>$n \times n$ matrices</td>
<td>“</td>
</tr>
<tr>
<td>polynomials</td>
<td>“</td>
</tr>
<tr>
<td>vectors</td>
<td>addition</td>
</tr>
<tr>
<td>$n$-bit strings</td>
<td>bitwise XOR</td>
</tr>
<tr>
<td>permutations over ${1, \ldots, n}$</td>
<td>composition</td>
</tr>
<tr>
<td>symmetries of a crystal</td>
<td>“</td>
</tr>
</tbody>
</table>

- In addition to studying these specific sets & operations individually, we identify general properties shared by many of them, such as:
  - commutativity: $a \cdot b = b \cdot a$
  - inverses (e.g. $-a$ for addition, $a^{-1}$ for multiplication)
  - unique factorization

- By abstracting such properties, algebra unifies our understanding of many disparate mathematical structures.

- Abstract algebra is useful in many science and engineering applications. Three that we will cover in this course:
  - Crystallography: the symmetry group of a crystal gives information about its physical properties.
  - Cryptography: encrypting data so that only the intended recipient can decrypt.
  - Error-correcting codes: encoding data so that it can be recovered from errors.

2 The Integers

- Reading: Gallian Chapter 0.

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1These notes are copied mostly verbatim from the lecture notes from the Fall 2010 offering, authored by Prof. Salil Vadhan. I will attempt to update them, but apologies if some references to old dates and contents remain.
• The integers are \( \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \).

• The natural numbers are \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

• Three (equivalent) forms of induction:
  
  - Well-ordering Principle: every nonempty subset of \( \mathbb{N} \) has a least element.
  
  - Standard Induction (Thm 0.4): if \( 0 \in S \) and for all \( n \in \mathbb{N} \) we have \( n \in S \Rightarrow n + 1 \in S \), then \( S \) contains all of \( \mathbb{N} \). (Induction can also be started at an arbitrary integer \( a \in \mathbb{Z} \) instead of 0; see text.)
  
  - Strong Induction (Thm 0.5): if \( 0 \in S \) and for all \( n \in \mathbb{N} \) we have \( \{0, \ldots, n\} \subseteq S \Rightarrow n + 1 \in S \), then \( S \) contains all of \( \mathbb{N} \).

• Induction usually formulated in terms of sequences of mathematical statements \( P(0), P(1), \ldots \), e.g. \( P(n) = "1 + \cdots + n = n \cdot (n + 1)/2" \). Correspondence to versions in terms of sets (Thms 0.4,0.5) is \( S = \{n : P(n) \text{ true}\} \).

• Proposition: For all \( n \in \mathbb{N} \), \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

  Proof by Induction:

• Thm 0.4: The Well-ordering Principle implies Standard Induction.

  Proof:

• Other directions are left as an exercise.