AM 106: Applied Algebra **Prof. Madhu Sudan**

Problem Set 4

Assigned: Wed. Sept. 27, 2017 Due: Sat. Oct. 7, 2017 (8:00 AM)

- You may submit your solutions via assignment page on the canvas website of the course.
- For collaboration and late days policy, see course website at `http://madhu.seas.harvard.edu/courses/Fall2017`
- Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Justify your answers except when otherwise specified.

Problem 1. (Orders of Permutations) What are all the possible orders for elements of S_8 and of A_8? (I.e., describe the set $S = \{i \in \mathbb{N} | \text{ there exists } a \in S_8 \text{ s.t. } |a| = i\}$. Similarly for A_8.) Justify your answers.

Problem 2. (Generating S_n) For a group G and elements $g_1, \ldots, g_n \in G$, the *subgroup generated by* $g_1, \ldots, g_n$ is defined to be the set of all elements we can obtain by multiplying the g_i's and their inverses together any number of times. Formally:

$$\langle g_1, \ldots, g_n \rangle = \left\{ g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_t}^{k_t} : t \in \mathbb{N}, i_1, \ldots, i_t \in \{1, \ldots, n\}, k_1, \ldots, k_t \in \mathbb{Z} \right\}.$$

(Note that a *cyclic* subgroup is a subgroup generated by a *single* generator g. Here we allow multiple generators, so these subgroups need not be cyclic.)

Prove that for $n \geq 2$, $S_n = \langle (12), (12 \cdots n) \rangle$. (Hint: repeatedly use conjugation to obtain all the transpositions.)

Problem 3. (Isomorphisms of Specific Groups) For each of the following pairs of groups (G, H), determine whether or not they are isomorphic. If not, determine whether one is isomorphic to a subgroup of the other. Justify your answers.

1. $\mathbb{Z}_5$ vs. S_5.
2. $\mathbb{Z}_8^*$ vs. $\mathbb{Z}_{12}^*$.
3. $\mathbb{R}^*$ vs. $\mathbb{C}^*$.

Problem 4. (From Cayley to Lagrange, Gallian 6.46)

1. Recall that in the proof of Cayley's Theorem, the isomorphism from a group G to a subgroup of $Sym(G)$ takes an element $g \in G$ to the permutation $T_g(x) = gx$. Show that for finite G, the disjoint cycle notation for T_g consists entirely of cycles of length equal to the order of g.

2. Deduce the following corollary of Lagrange's Theorem: the order of an element $g \in G$ divides the order of the group G.