Problem 1. (Orders of Permutations) What are all the possible orders for elements of $S_8$ and of $A_8$? (I.e., describe the set $S = \{ i \in \mathbb{N} \mid \text{there exists } a \in S_8 \text{ s.t. } |a| = i \}$. Similarly for $A_8$.) Justify your answers.

Problem 2. (Generating $S_n$) For a group $G$ and elements $g_1, \ldots, g_n \in G$, the subgroup generated by $g_1, \ldots, g_n$ is defined to be the set of all elements we can obtain by multiplying the $g_i$’s and their inverses together any number of times. Formally:

$$\langle g_1, \ldots, g_n \rangle = \left\{ g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_t}^{k_t} : t \in \mathbb{N}, i_1, \ldots, i_t \in \{1, \ldots, n\}, k_1, \ldots, k_t \in \mathbb{Z} \right\}.$$  

(Note that a cyclic subgroup is a subgroup generated by a single generator $g$. Here we allow multiple generators, so these subgroups need not be cyclic.)

Prove that for $n \geq 2$, $S_n = \langle (12), (12 \cdots n) \rangle$. (Hint: repeatedly use conjugation to obtain all the transpositions.)

Problem 3. (Isomorphisms of Specific Groups) For each of the following pairs of groups $(G, H)$, determine whether or not they are isomorphic. If not, determine whether one is isomorphic to a subgroup of the other. Justify your answers.

1. $\mathbb{Z}_5$ vs. $S_5$.
2. $\mathbb{Z}_6$ vs. $\mathbb{Z}_{12}$.
3. $\mathbb{R}^*$ vs. $\mathbb{C}^*$.
Problem 4. (From Cayley to Lagrange, Gallian 6.46)

1. Recall that in the proof of Cayley’s Theorem, the isomorphism from a group $G$ to a subgroup of $Sym(G)$ takes an element $g \in G$ to the permutation $T_g(x) = gx$. Show that for finite $G$, the disjoint cycle notation for $T_g$ consists entirely of cycles of length equal to the order of $g$.

2. Deduce the following corollary of Lagrange’s Theorem: the order of an element $g \in G$ divides the order of the group $G$. 