Problem 1. (Homomorphisms) Give three distinct homomorphisms from \( \mathbb{Z}_{30} \) to \( \mathbb{Z}_{84} \). For each, identify the kernel and image. (Hint: how does \( \varphi(x) \) relate to \( \varphi(1) \)? And what can we say about the order of \( \varphi(1) \)?)

Problem 2. (Normal Subgroups of Small Index)

1. Show that if \( H \) is a subgroup of \( G \) of index 2, then \( H \) is normal in \( G \).
2. (Optional, 0 points) Show that if \( G \) has no subgroups of index 2, then every subgroup of index 3 is normal. (Hint: Multiplication on the left by \( g \in G \) permutes the cosets of \( H \). This gives rise to a homomorphism \( \varphi : G \to S_3 \). Reason about \( \text{Ker}(\varphi) \), \( \text{Im}(\varphi) \), and \( \varphi^{-1}(A_3) \).)

Problem 3. (Cautions with Normality and Factor Groups)

1. Give an example of a group \( G \) and normal subgroup \( N \) such that \( G/N \times N \not\cong G \).
2. Give an example of a group \( G \) and subgroups \( N \) and \( H \) such that \( H \triangleleft N \triangleleft G \), but \( H \not\triangleleft G \). Justify your answers.

Problem 4. (Factor Groups and Homomorphisms) For each of the following groups \( G \) and subsets \( H \subseteq G \), determine whether \( H \) is a normal subgroup of \( G \). If yes, then find a familiar group \( G' \) such that \( G/H \cong G' \). Prove that \( G/H \cong G' \) by giving an appropriate homomorphism from \( G \) to \( G' \).

1. \( G = \mathbb{Z}, H = \{ \text{prime integers} \} \).
2. \( G = S_5 \times S_5, H = \{ (\sigma, \sigma) : \sigma \in S_5 \} \).
3. \( G = \mathbb{C}^*, H = S^1 = \{ z \in \mathbb{C}^* : |z| = 1 \} \).