Problem 1. (From Translations and Point Groups to the Full Symmetry Group) Let $E_2$ be the 2-dimensional Euclidean group, i.e., the group of isometries in $\mathbb{R}^2$ under composition. Let $F : X \rightarrow \mathbb{R}^2$ be a 2-dimensional crystal.

1. Let $E_2^+$ denote the set of rotations in $E_2$, i.e. the set of isometries of the form $T(x) = \text{Rot}_\theta x + b$, for $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^2$. Show that $E_2^+$ is a subgroup of $E_2$, and that it is of index 2.

2. Let $\text{Isom}(F)^+ = \text{Isom}(F) \cap E_2^+$. Show that either $\text{Isom}(F)^+ = \text{Isom}(F)$ or $\text{Isom}(F)^+$ is a subgroup of $\text{Isom}(F)$ and that it is of index 2. Similarly, for a point $p \in \mathbb{R}^2$, if we define $\text{Point}(F, p)^+ = \text{Point}(F, p) \cap E_2^+$ then $\text{Point}(F, p)^+$ either equals $\text{Point}(F, p)$ or is a subgroup of $\text{Point}(F, p)$ of index 2. (Hint: these statements are have nothing to do with geometry, and generalize to studying the intersection $H^+$ of arbitrary subgroups $G^+, H$ of a group $G$ such that $[G : G^+] = 2$.)

3. Let $\text{Rot}(F) = \{\text{Rot}_\theta : \exists b \text{ s.t. } T(x) = \text{Rot}_\theta x + b \text{ is in } \text{Isom}(F)\}$. Show that $\text{Rot}(F)$ is a cyclic group generated by $\text{Rot}_{\theta^*}$ for the smallest positive value of $\theta^*$ such that $\text{Rot}_{\theta^*} \in \text{Rot}(F)$.

4. Prove that if $p$ is taken to be a point of highest rotational symmetry, then

$$\text{Isom}(F)^+ = \{T_1 \circ T_2 : T_1 \in \text{Trans}(F), T_2 \in \text{Point}(F, p)^+ \} \overset{\text{def}}{=} \text{Trans}(F) \circ \text{Point}(F, p)^+.$$ (For notational simplicity, you may take assume that $p = 0$.)

5. Deduce that if $p$ is a point of highest rotational symmetry, then one of the following cases must hold:

(a) $\text{Isom}(F)$ does not contain a reflection or glide-reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$.

(b) $\text{Point}(F, p)$ contains a reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$. 

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(c) Isom($F$) contains a reflection or glide-reflection $R$, Point($F,p$) does not contain a reflection, and Isom($F$) = (Trans($F$) o Point($F,p$)) $\cup$ (Trans($F$) o Point($F,p$) o $R$).

In particular, we can obtain generators for Isom($F$) by taking generators for Point($F,p$) (at most 2 needed), generators for Trans($F$) (exactly 2 needed), and possibly an additional reflection $R$.

**Problem 2. (Characteristic and Order of Finite Fields)**

1. Show that if $R$ is an integral domain of nonzero characteristic $p$, then every nonzero element of $R$ has additive order $p$.

2. Use the classification of finite abelian groups to show that if $F$ is a finite field of characteristic $p$, then the order (i.e. size) of $F$ is $p^n$ for some $n \in \mathbb{N}$.

**Problem 3. (Adjoining Square Roots)** Which of the following rings are integral domains? fields? Justify your answers.

1. $\mathbb{Z}_{15}[\sqrt{2}]$. (Elements are of the form $(a + b\sqrt{2})$ with $a,b \in \mathbb{Z}_{15}$, addition defined by $(a + b\sqrt{2}) + (c + d\sqrt{2}) = ((a + c) \mod 15) + ((b + d) \mod 15)\sqrt{2}$, and multiplication defined by $(a + b\sqrt{2})(c + d\sqrt{2}) = ((ac + 2bd) \mod 15) + ((ad + bc) \mod 15)\sqrt{2}$.)

2. $\mathbb{Z}_{11}[\sqrt{2}]$. (Defined similarly to previous item.)

3. $\mathbb{Z}_{7}[\sqrt{2}]$. (Defined similarly to previous item.)

4. (Optional, 0 points) Characterize when $\mathbb{Z}_n[\sqrt{k}]$ is a field for arbitrary positive integers $n$ and $k$. Your characterization should take the form of “$\mathbb{Z}_n[\sqrt{k}]$ is a field if and only if $n$ has Property X and the equation ‘$\cdots = \cdots$’ (in one variable $x$) has no solution in $\mathbb{Z}_n$.”