CS 121: Lecture 13
Turing Equivalence & Universality

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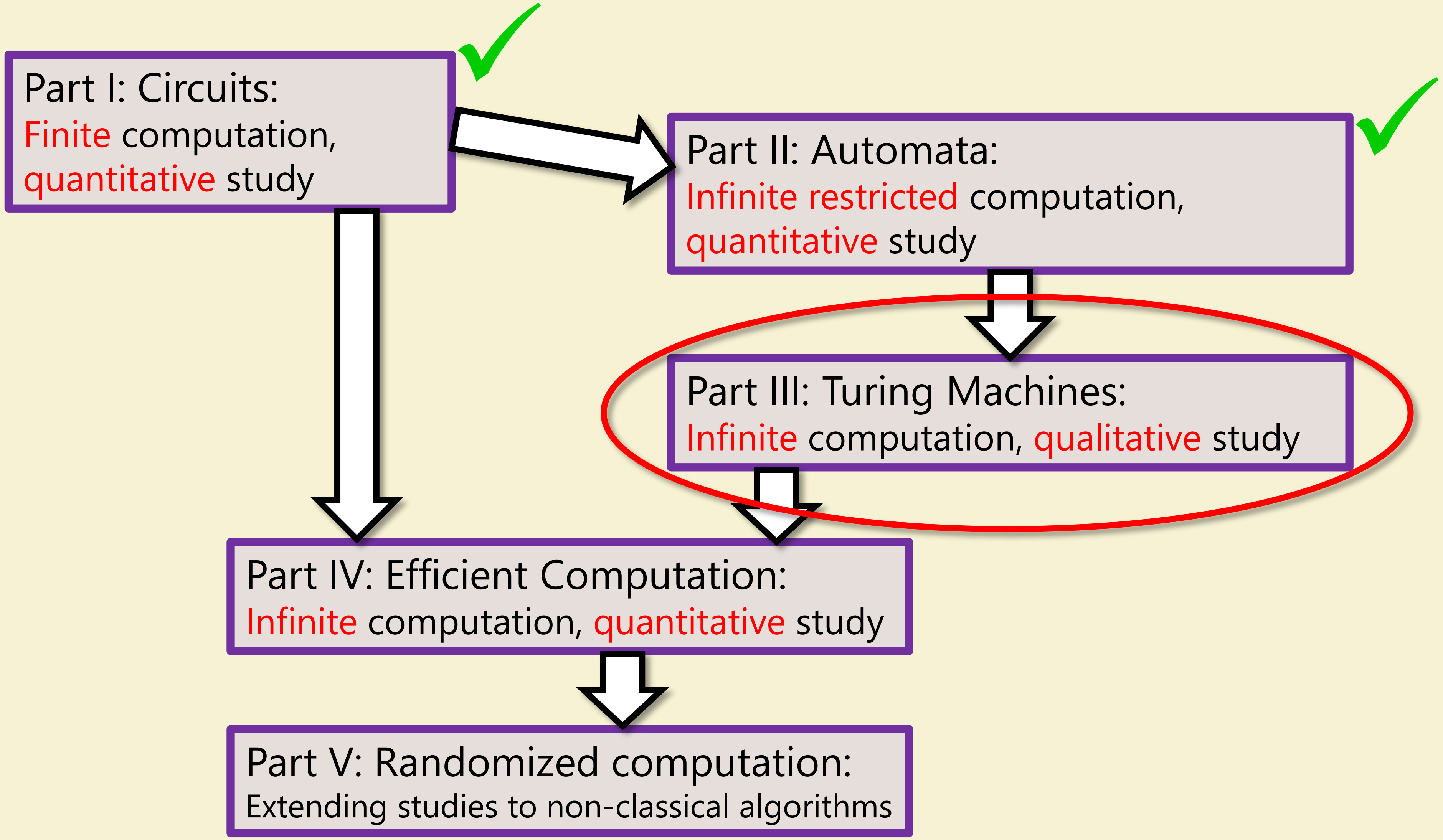
Book: https://introtcs.org

How to contact us
- The whole staff (faster response): CS 121 Piazza
- Only the course heads (slower): cs121.fall2020.course.heads@gmail.com

Announcements:

- Advanced Sections: Josh Alman on Matrix Multiplication
- Midterms yet to be graded. Will post details on Piazza when ready
- Homework 4 out today. Due in two weeks.
- Participation Survey done?
  - Sign up for active participation here!
- Midterm Feedback Survey coming soon!
  - Mandatory (5 points on homework 4.). Anonymous!
  - Staff takes it seriously! (Be open – call out specific people, actions).
- Section 6 cycle starts today. Material in usual place!

Where we are:


Part I: Circuits:
Finite computation, quantitative study
✓
Part II: Automata:
Infinite restricted computation, quantitative study
✓
Part III: Turing Machines:
Infinite computation, qualitative study
Part IV: Efficient Computation:
Infinite computation, quantitative study
Part V: Randomized computation:
Extending studies to non-classical algorithms

Today:

- Two results to be aware of, and to use (heavily)?
- No proofs to know/remember.
    - Proofs/sketches available in book.
    - We will discuss. But suffices to know they exist!
- Result 1: Turing-Church Thesis
    - Provable part: TMs as powerful as any high-level programming language.
    - Usable part: To prove computability, suffices to give program in high-level lang.
- Result 2: ∃ a Universal Turing Machine
    - Takes as input description $E(M) \in \{0,1\}^*$ of any Turing Machine, and $x \in \{0,1\}^*$
    - Outputs $M(x)$, the result computed by M on x (if M halts) – no output otherwise.

Recall Turing Machines

- (Barak, Definition 7.1):
- TM on k states and alphabet $\Sigma \supseteq \{0,1,\triangleright,\phi\}$

  is given by $\delta\colon [k] \times \Sigma \to [k] \times \Sigma \times \mathrm{Action}$,

  where $\mathrm{Action} = \{L, R, S, H\}$
  - L=Left, R=Right, S=Stay (don't move), H=Halt (done!!)
- Operation:
  - Start in state 0, Tape $T =$ ▯$x_0 \dots x_{n-1}\phi\phi\phi \dots$, Head (i) at x_0
  - General step: current state q ; input symbol σ:

    Let $\delta(q,\sigma) = (r,\tau,X) \Rightarrow$ Write τ on tape (overwriting σ) ; Move to state r;
    Move Head left ($i \leftarrow i - 1$) if $X = L$; right if $X = R$; don't move if $X = S$.
  - Repeat General step until $X = H$

Exercise Break 1

- Turing M → No random access.

- Pick a high-level language
- Identify features that are very different from Turing Machines.
- Discuss differences after the break.

Array

- Ocaml → Recursion
- Python → Classes, Objects (Type checking)
- Python → Dictionaries, Stacks, Adv. Data Structures

My list of differences:

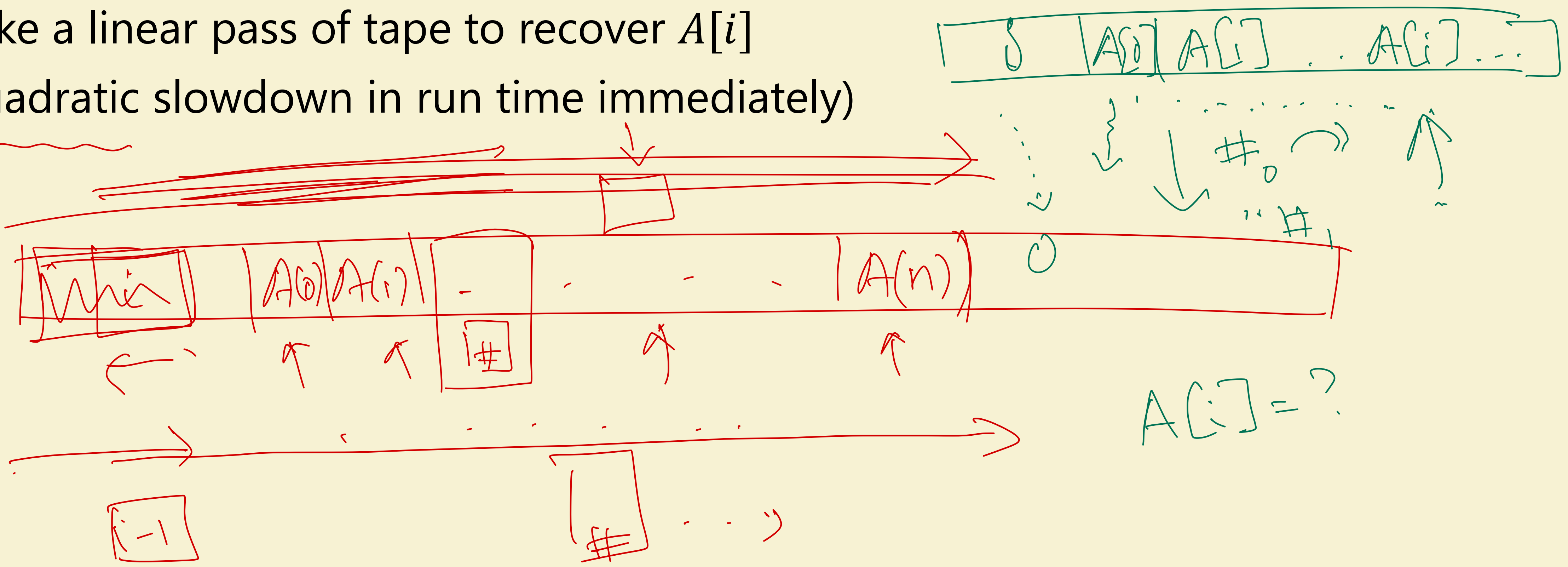
- General programming languages allow multiple, multidimensional arrays!
  - TMs have one array : $\mathrm{Tape}[0, \infty]$
- Allow ``random” (arbitrary) access into arrays/memory.
  - Can look at $A[i]$ in one step and then $A[i^2 + 10i + 5]$ or even $A\left[A[i]\right]$ in next step
  - TMs: If this step involves $\mathrm{Tape}[i]$

    then next can only involve $\{\mathrm{Tape}[i-1], \mathrm{Tape}[i], \mathrm{Tape}[i+1]\}$
- Rest? Syntactic Sugar
  - Sophisticated constructs: loops, cases, recursion
  - Data structures: Lists, Queues, Stacks …

Dealing with the differences - 1

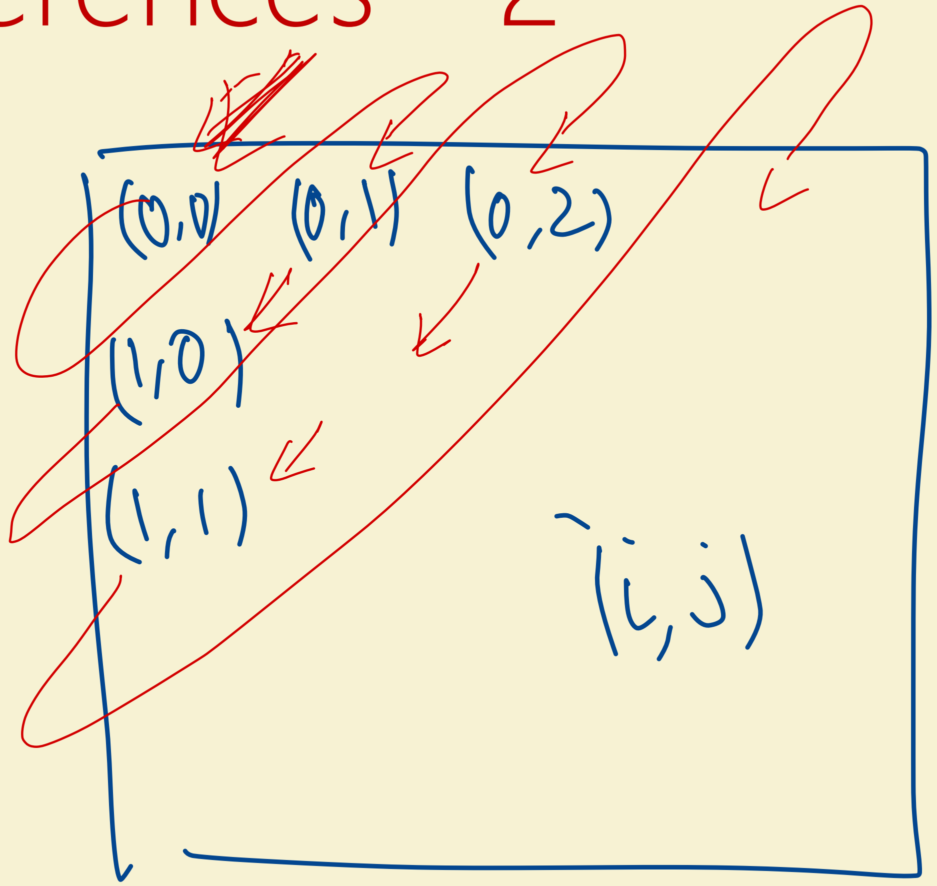
$A[A[A[i]]]$

- Random access:
  - Deal with by brute force.
  - Store index on Tape. Compute new index and overwrite on tape.
  - Make a linear pass of tape to recover $A[i]$
  - (Quadratic slowdown in run time immediately)



Dealing with the differences - 2

- Multiple Arrays+Indices
    - Same solution.
- Multi-dimensional Arrays
    - (Draw this out)



- Consequence: If algorithm A runs in time T with high-level program, can be implemented to run in time $O(T^2)$ on Turing Machine.
- Details in Barak: Chapter 8

Road Map of details

- TMs
- Define NAND-TMs. Show equivalent to TMs.
  - Just a program version of TMs. Like NAND circuits vs. NAND-CIRC programs.
- Define NAND-RAMs. Show equivalent to NAND-TMs.
  - Allows loops and general indices.
  - This is the crucial step.
- Define RAM machines. Show equivalent to NAND-RAMs
  - This what most compilers use to compile "down" from the high-level spec.
  - Equivalence straightforward.

“HOCAEIT” Theorem

Have Our Cake And Eat It Too

- Recall definition of **Computable**.
  - $F : \{0,1\}^* \rightarrow \{0,1\}^*$ is computable iff it is computable by TM.
- **Equivalence (HOCAEIT) Theorem:** TMs are equivalent to High-Level Languages.
- Having our cake: To prove F is computable only need to exhibit algorithm in high-level language.
- Eating it: To prove F is not computable only need to rule out TMs.

Church-Turing Thesis

- “Every function that is computable by physical means is (Turing Machine) computable.”
- Some (made-up?) history:
    - Church defined computability with λ-calculus
    - Turing + Church compared notes and agreed their models were equivalent.
    - Many other models were shown to be equivalent.
    - Turing went on to do a postdoc under von Neumann.
    - Von Neumann later introduced the “stored program architecture” of computer to the computer architects of the time. Led to the first physical computers.
    - Conway invented Game of Life … simplest Turing Equivalent model?

Universality

- "One machine to rule them all"
- "There exists a single program/algorithm/TM that can run all other programs/algorithms/TMs."

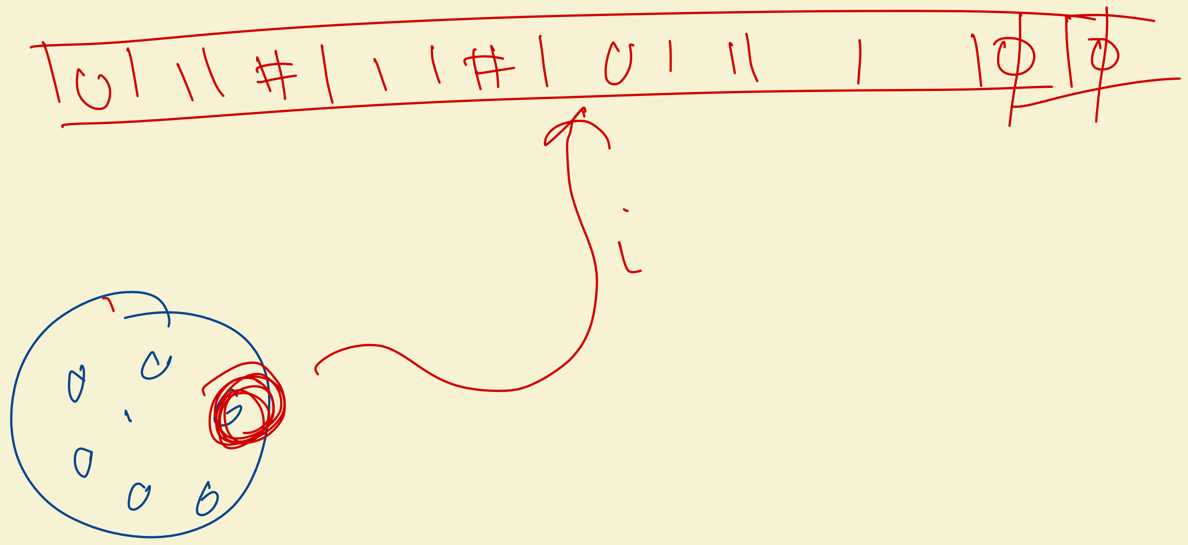
- Formally:
  1. There exists a way to encode Turing Machines so that they can be (part of) input to other Turing Machines.
  2. The exists a universal machine U that takes as input a pair (M, x) and outputs $U(M, x) = M(x)$ (if M halts on x)

Part 1: Encoding Turing Machines

- Should be familiar to us:
- Recall M specified by $\Sigma \supseteq \{>, 0,1, \phi\}$, k, $\delta: [k] \times \Sigma \rightarrow [k] \times \Sigma \times \{L, R, S, H\}$
  - First encode $E_\Sigma: \Sigma \rightarrow \{0,1\}^c$; $E_A: \{L, R, S, H\} \rightarrow \{0,1\}^2$, $E_k: [k] \rightarrow \{0,1\}^{\log k}$ so $\delta: \{0,1\}^{\log k + c} \rightarrow \{0,1\}^{\log k + c + 2}$
  - Encoding of $M = \text{Enc}\big(c, k, \delta(0,000), \delta(0,001) \dots \delta(k-1,111)\big)$
  - Where $\text{Enc}: \mathbb{N} \times \mathbb{N} \times \left(\{0,1\}^{\log k + c + 2}\right)^{k2^c} \rightarrow \{0,1\}^*$ is some 1-1 function.
  - Encoding of M $= \text{Enc}(c, k, \delta)$

Part 2: Interpreting the Encoding

- Definition: Configuration of a machine M on input x after t steps of computation, denoted C_t, is the "full state of the computation":
  - Current state of Turing Machine
  - Current contents of the Tape
  - Current location i of Tape head



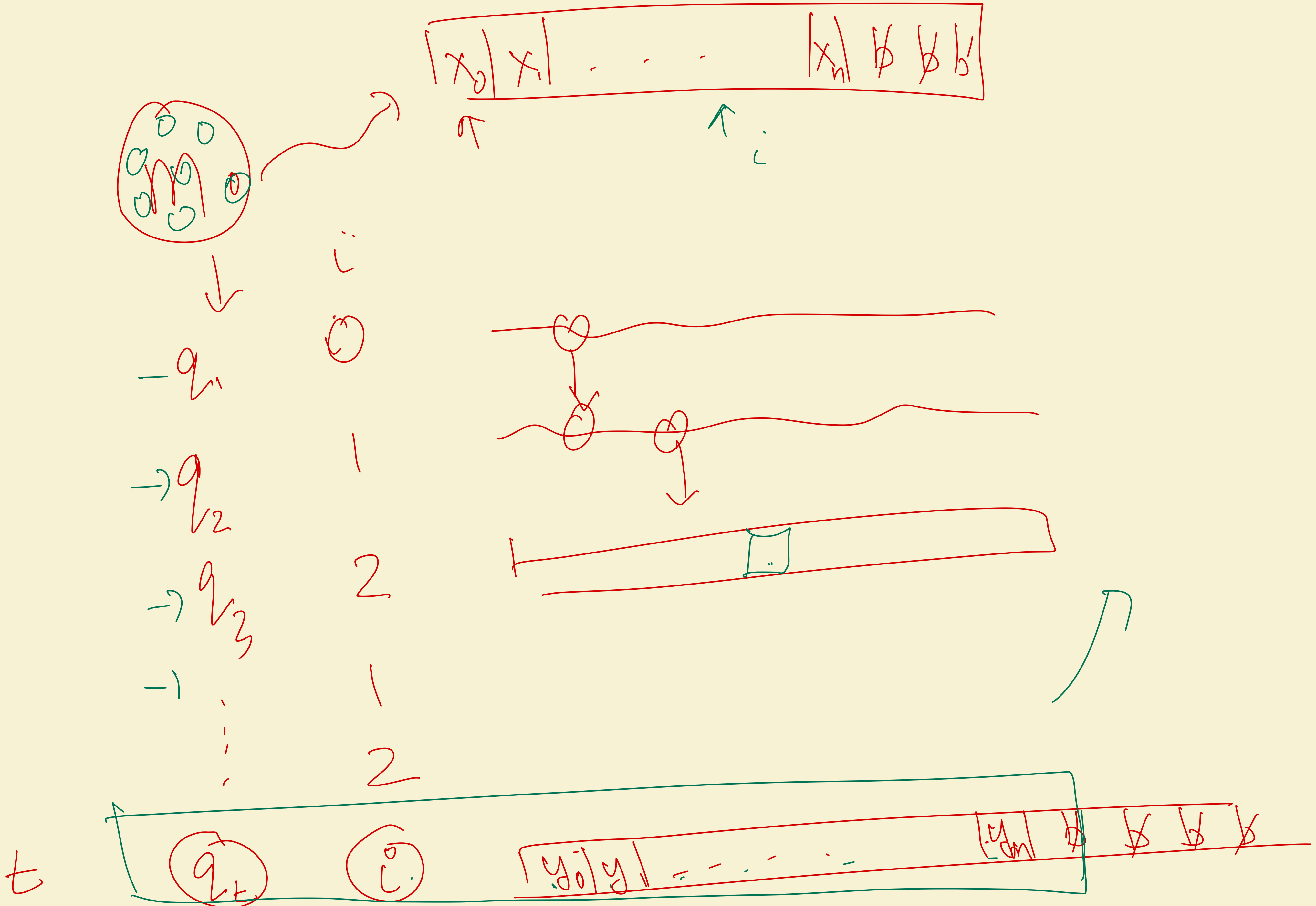
- Core of Universal TM U
  - "Universal-Stepper": $(M, C_t) \mapsto (M, C_{t+1})$

Exercise Break 2

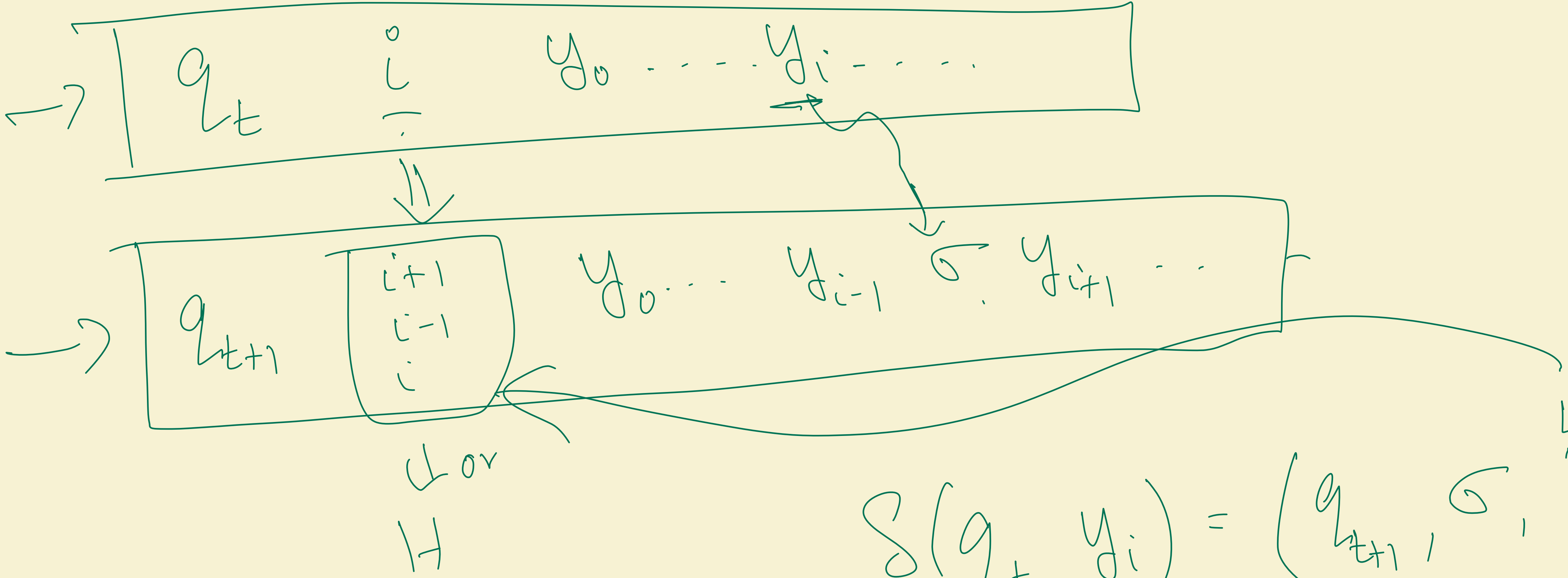
Definition: Configuration of a machine M on input x after t steps of computation, denoted C_t, is the “full state of the computation”:

- Current state of Turing Machine
- Current contents of the Tape
- Current location i of Tape head

- Discuss how to organize the information (M, C_t) on U's tape:
- Describe (in English) steps needed to compute $(M, C_t) \mapsto (M, C_{t+1})$

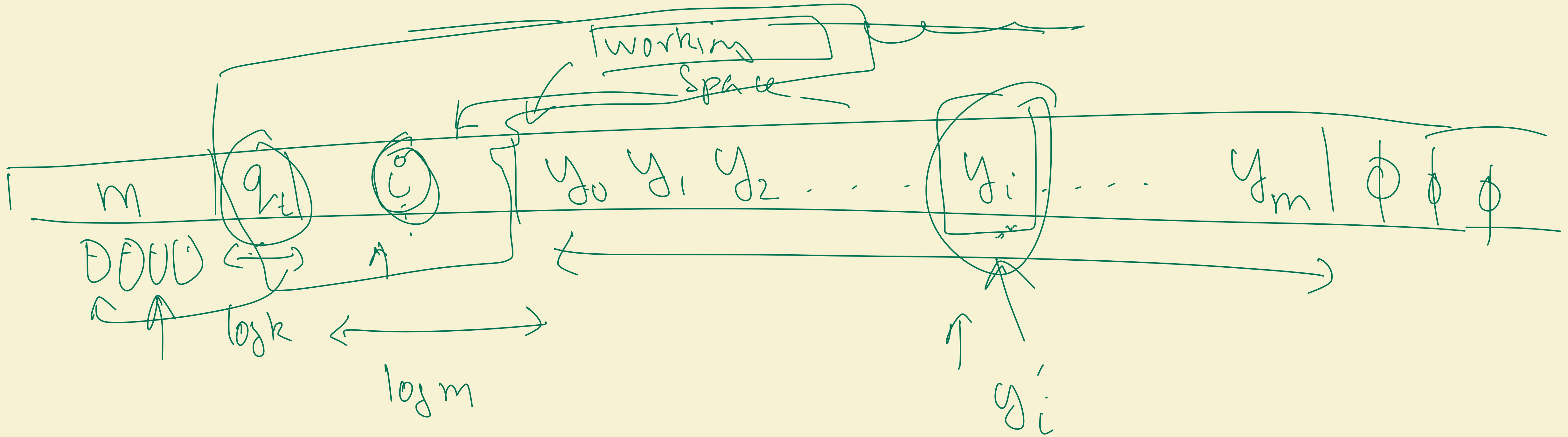


x_0 x_1 . . . x_n b b b
i
q_1
q_2
q_3
i
0
1
2
1
2
t
q_t
i
y_0 y_1 . . . y_m b b b b



$$\delta(q_t, y_i) = \left(q_{t+1}, \sigma, \begin{array}{c} L \\ R \\ H \\ S \end{array}\right)$$

Computing $(M, C_t) \mapsto (M, C_{t+1})$



- Initially: Make space for (current state, head location, current symbol)
- In each round:
  - fetch contents of Tape[head location] and update
  - Look at the code of the TM to determine next state, next location, symbol to write.
  - Write the "symbol to write" at current location.
  - Update "head location"
- Conclusion: Lots of string manipulation (string copy), adjust … nothing profound.

$\delta(q_t, i) = ?$

Summary of Lecture:

- Turing Equivalence and Turing-Church Thesis:
  - No proofs to remember. But encouraged to read the text (Chapter 8)
  - Do remember the HOCAEIT theorem! “Do not leave home without it.”
    - To prove computability, give algorithm in high-level language.
    - To prove non-computability, rule out TMs.
- $\boxed{\text{Universal Turing machines:}}$
  - Single machine to simulate all others:
    - Similar to circuits.
    - Big difference: Simulates larger machines over larger alphabets!!!!

Next Lecture

- Uncomputability.
  - Some functions are not computable no matter how much time we are willing to take!