CS 121: Lecture 7
Infinite Functions

AND FINITE AUTOMATA

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- The whole staff (faster response): CS 121 Piazza
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Reminders

- 121.5: Ryan O’Donnell, Analysis of Boolean Functions. Today @ 4:30
- Section 3 cycle begins today
- Extra-length sections:
  - Will (Thursdays: 6-7:30pm), Max & Zuzanna (Tuesdays: 7:30am-9am).

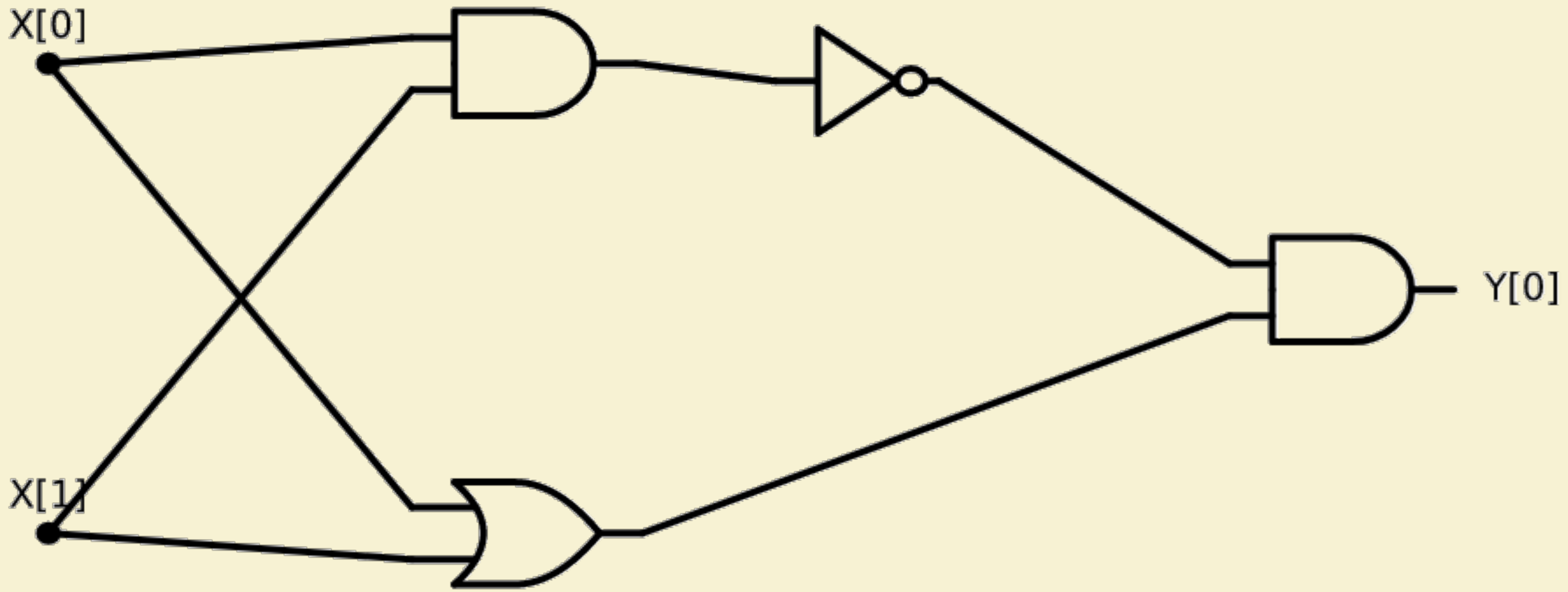
Today

- Infinite vs. Finite functions
- Example: Addition as finite state algorithm
- (Deterministic) Finite Automata:
- Break 1: Understand DFA
- Break 2: Design DFA
- Preview of next lecture: Regular Expressions

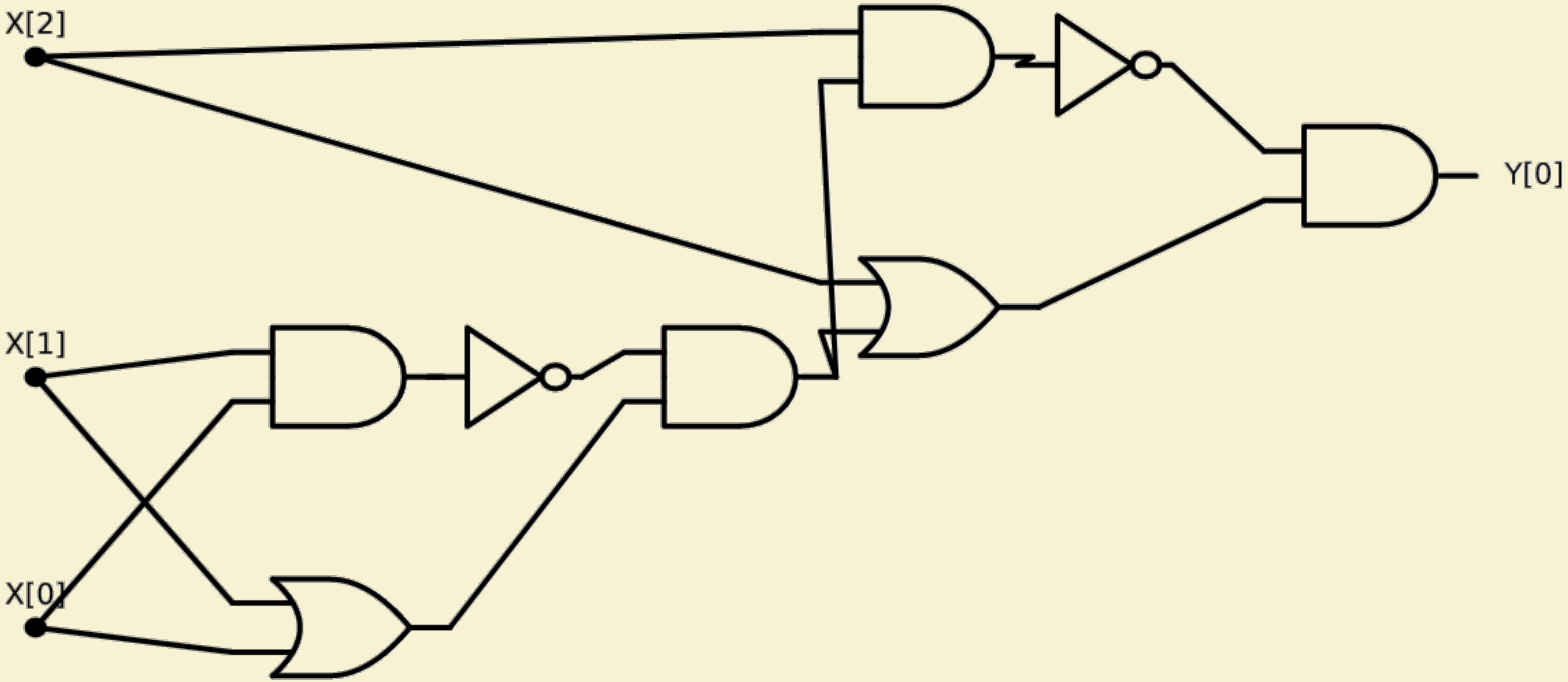
So far

- Have seen Circuits/NAND-CIRC Programs
- Compute all finite functions:
  - Given $f: \{0,1\}^n \rightarrow \{0,1\}^m$, exists NAND-CIRC C, s.t. $\forall x \in \{0,1\}^n, C(x) = f(x)$
  - Sounds great?

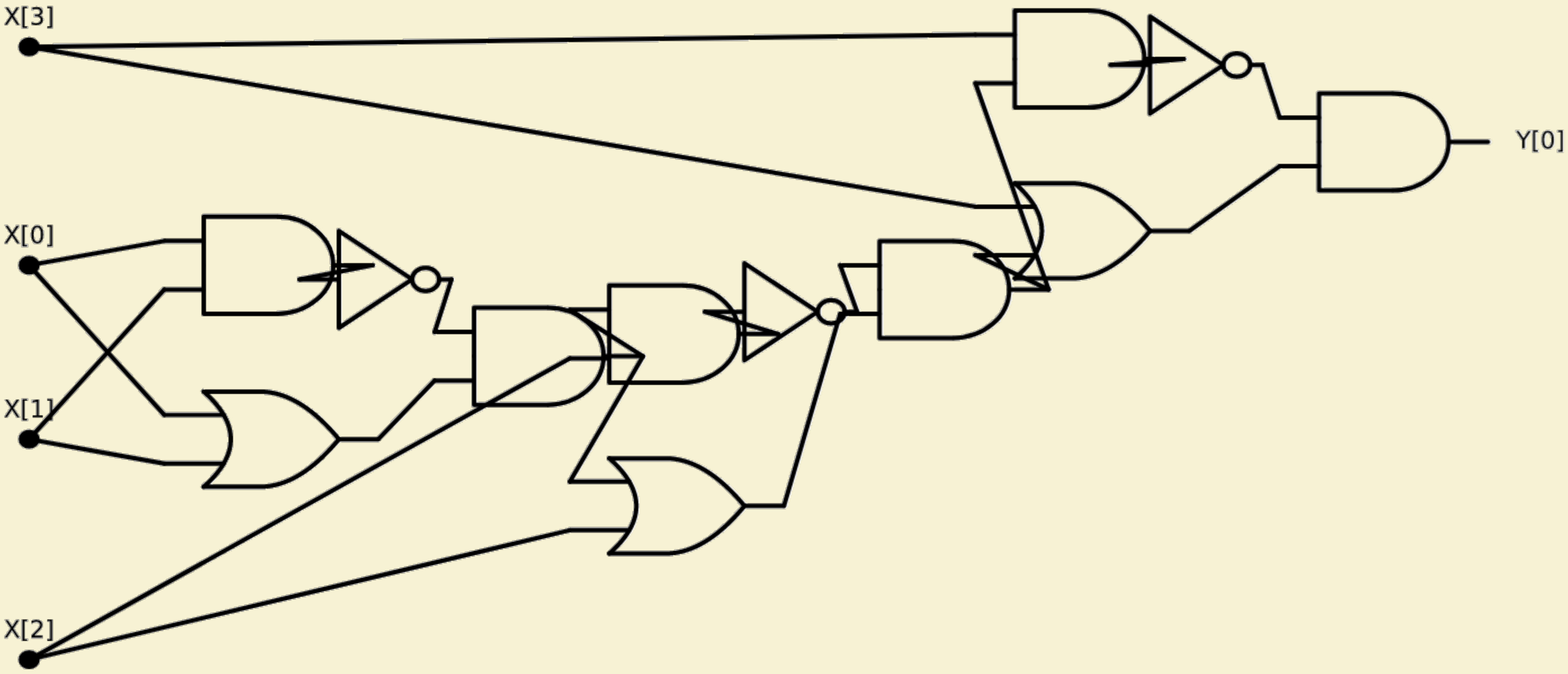
XOR on 2 variables



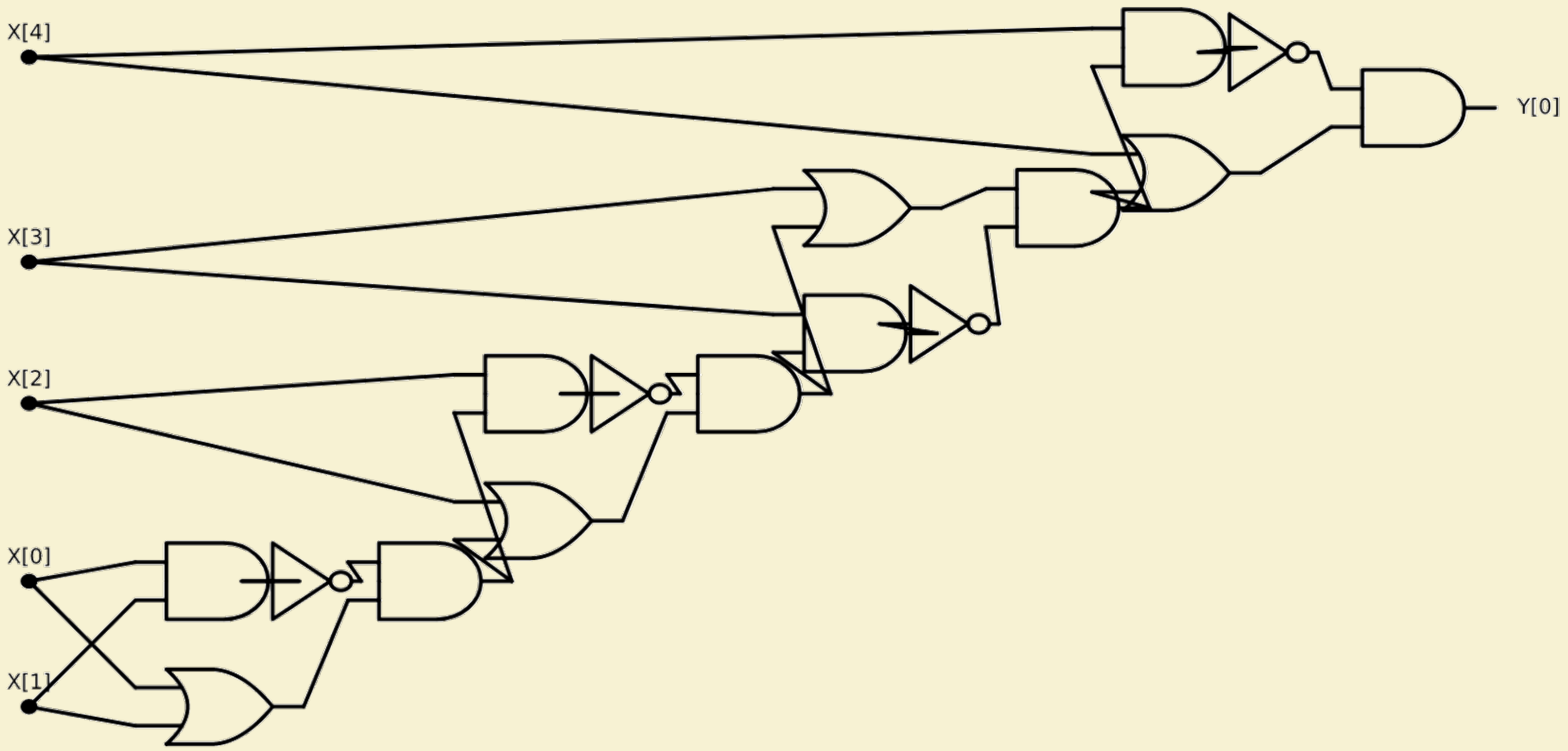
XOR on 3 variables



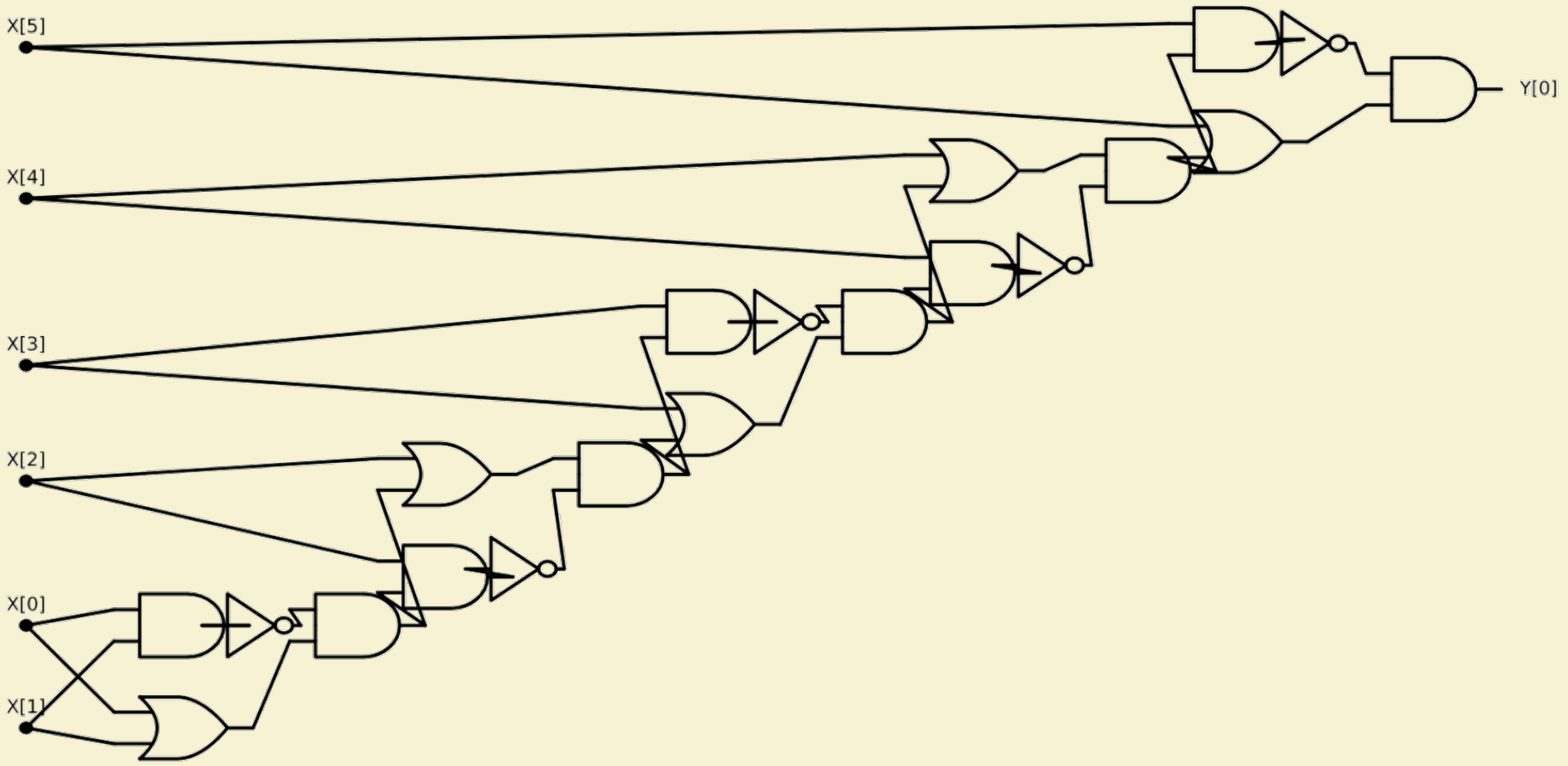
XOR on 4 variables



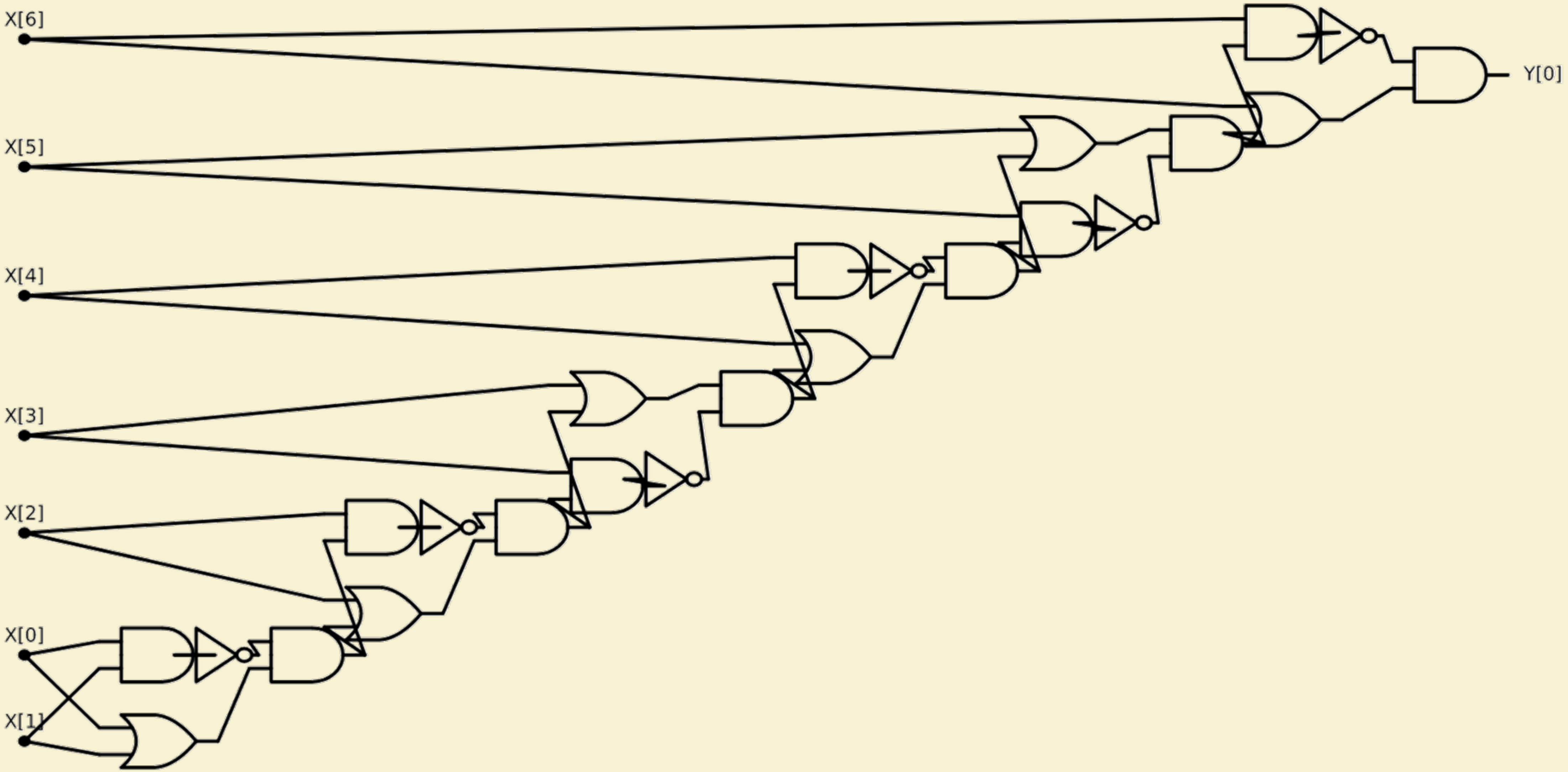
XOR on 5 variables



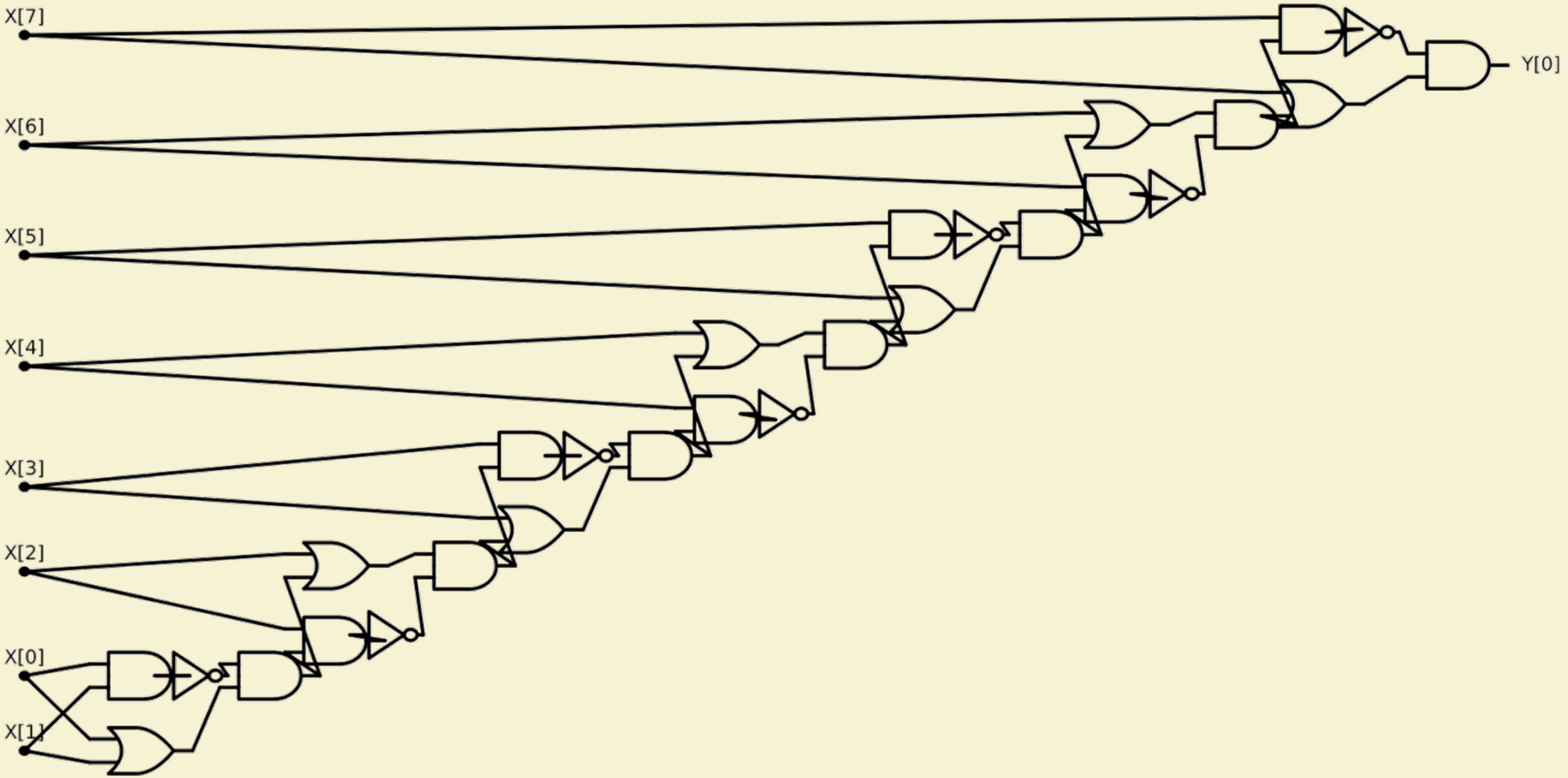
XOR on 6 variables



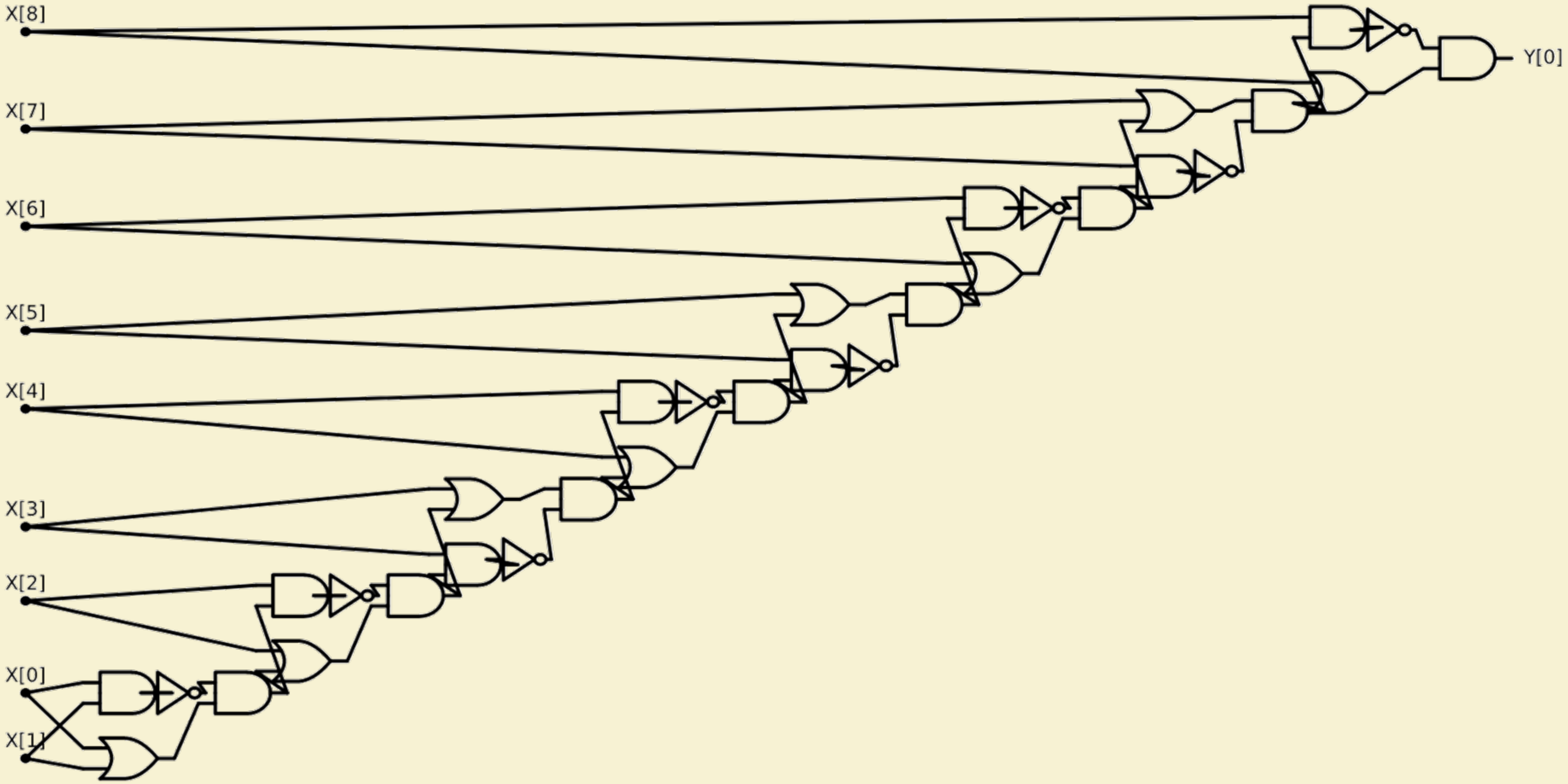
XOR on 7 variables



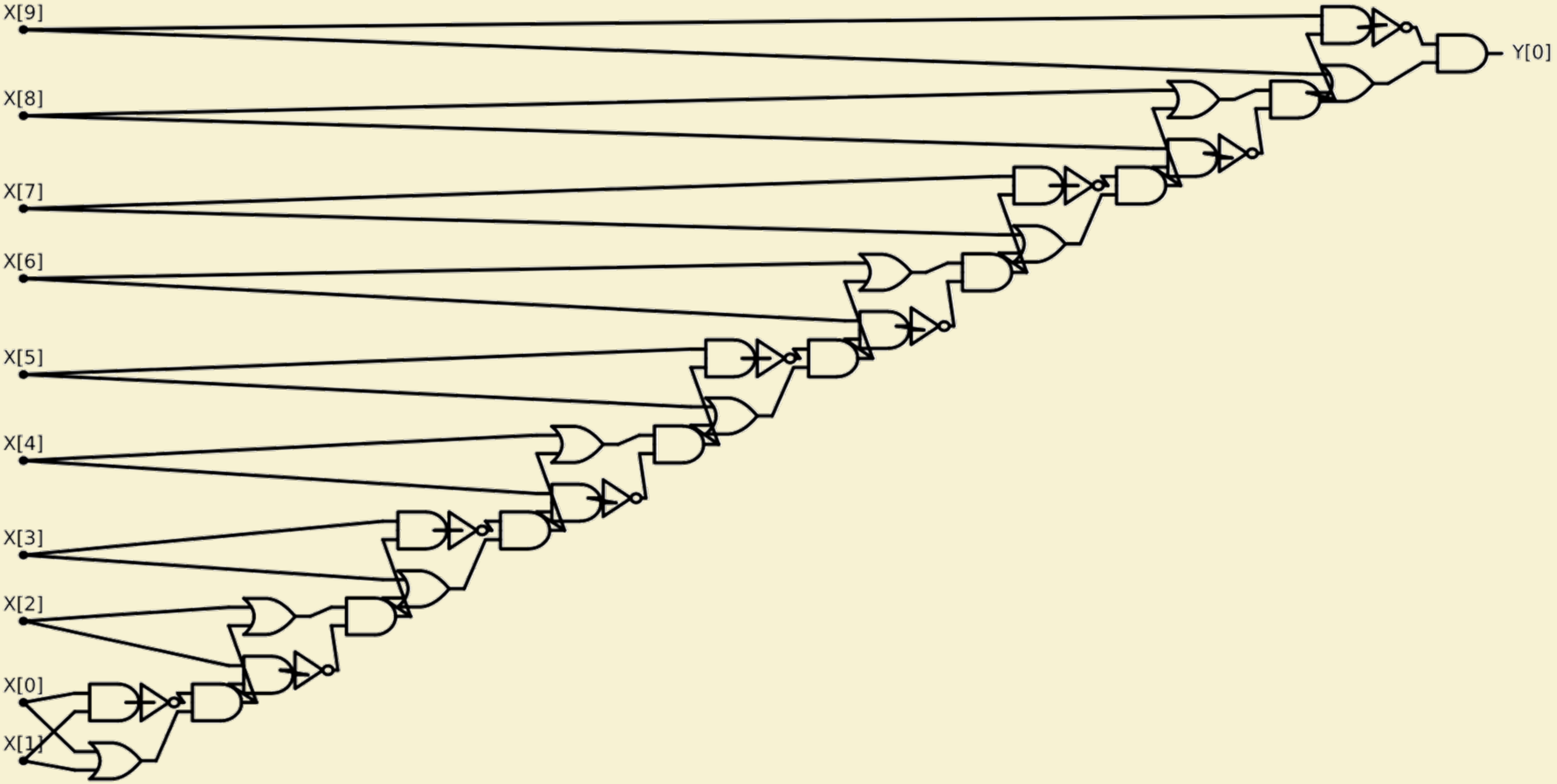
XOR on 8 variables



XOR on 9 variables



XOR on 10 variables



So far

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- Compute all finite functions:
  - Given $f: \{0,1\}^n \to \{0,1\}^m$, exists NAND-CIRC C, s.t. $\forall x \in \{0,1\}^n, C(x) = f(x)$
  - Sounds great?

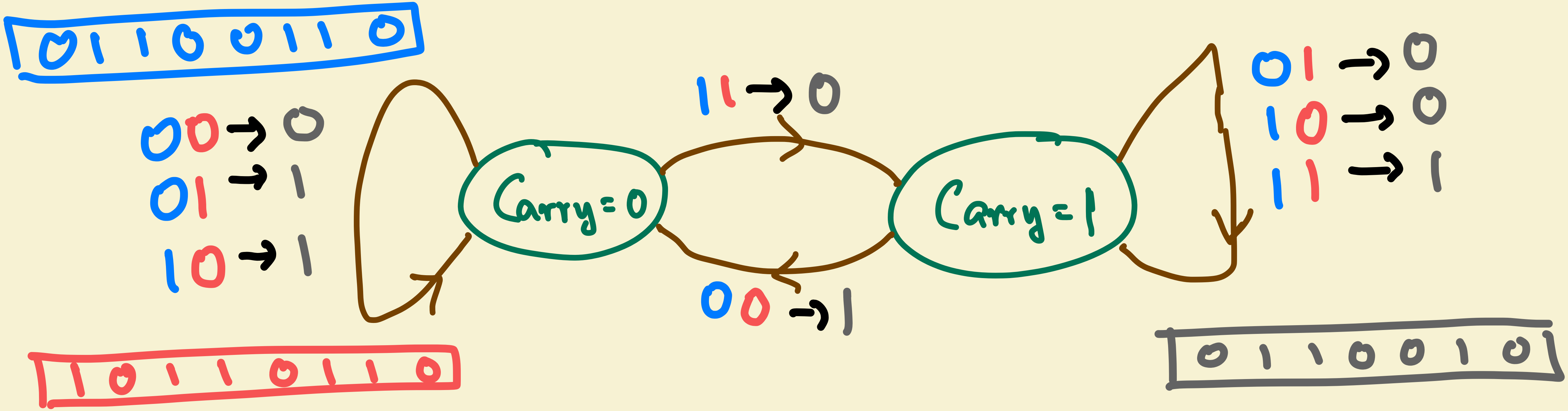
- Weakness: Only computes finite functions.
  - No generalization?
  - Given circuits for $\mathrm{ADD}_1, \mathrm{ADD}_2, \mathrm{ADD}_3, \ldots \mathrm{ADD}_n$ - do we know what circuit for ADD_{n+1} looks like?
  - Our favorite algorithms generalize!!

Today: Algorithms with finite state

- What should an algorithm be?
  - Sequence of simple steps
  - Different steps for different inputs
  - Exactly which step to take must be determined (based on input, easily, locally).
  - Different #steps for different input lengths
  - When to stop must be determined (based on input, easily, locally).

- Finite state algorithms: What step to take, when to stop determined by "finite state" (constant # bits of memory).

Example: Addition as finite state algorithm



- Advantage: O(1)-sized description. Tells how to compute an infinite function $\mathrm{ADD}: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$
- Can you do anything else?
  - Multiplication? NO ☹
  - … but can do modular counting, pattern matching

Boolean functions

- From now will focus only on Boolean functions: $G: \{0,1\}^* \to \{0,1\}$
- Why?
  - Given $F: \{0,1\}^* \to \{0,1\}^*$, can design $\mathrm{b}F: \{0,1\}^* \times \mathbb{N} \to \{0,1\}$ or $BF: \{0,1\}^* \times \overset{\{0,1\}^*}{\cancel{\mathbb{N}}} \to \{0,1\}$, that are roughly "equally easy/hard".
  - Idea: $BF(x, i) = F(x)_i$
  - If $F(x) \in \{0,1\}^m$ for some m:
    - Can go from $F(x)$ to $BF(x, i)$ (for any single i) by erasing other parts of output.
    - Can go from $BF(x, i)$ to $F(x)$ by m calls to algorithm for $BF(\cdot,\cdot)$

Exercise Break 1

- Booleanize $Mult: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$, where $Mult$ is the multiplication function for integers (given in "little-endian").

$BMult(x, y, i) = (x \cdot y)_i$

- What is domain of your function?
- What is the range?

$\{0, 1, \perp\}$

Suppose: $Mult(x, y) \in \{0,1\}^m$

$BMULT(x, y, 2m) = ?$

if $f(x, y) \in \{0,1\}^m \Rightarrow$ if $i \in [m]$
$Bf(x, i) = f(x)_i$

Deterministic Finite Automata (DFA)

- Finite algorithms computing Boolean functions: $f: \{0,1\}^* \to \{0,1\}$
- Operation:

$E: \{\text{Transition functions}\} \to \{0,1\}^*$ $O(c \log c)$

$E(T \text{ on } C \text{ States}) \in \{0,1\}^*$

  1. Finite number of states: C
  2. Starts in state 0, reads x_0
  3. At any stage has current state q, last read input symbol σ
  4. Moves to state $T(q,\sigma)$; moves to read next input symbol
  5. If input not done, repeat from Step 3.
  6. When done: Accept (output 1) if current state $q \in S$ and reject (output 0) otherwise.

E is 1-1.

- Specification:, (T,S) where $T:[C] \times \{0,1\} \to [C]$, $S \subseteq$ $[C]$
- (more elaborate spec. in Sipser): (Q, q_0, Σ, T, S) [$Q = [C], q_0 = 0, \Sigma = \{0,1\}$]

Example:

$T = ?$ $\qquad C = 4$

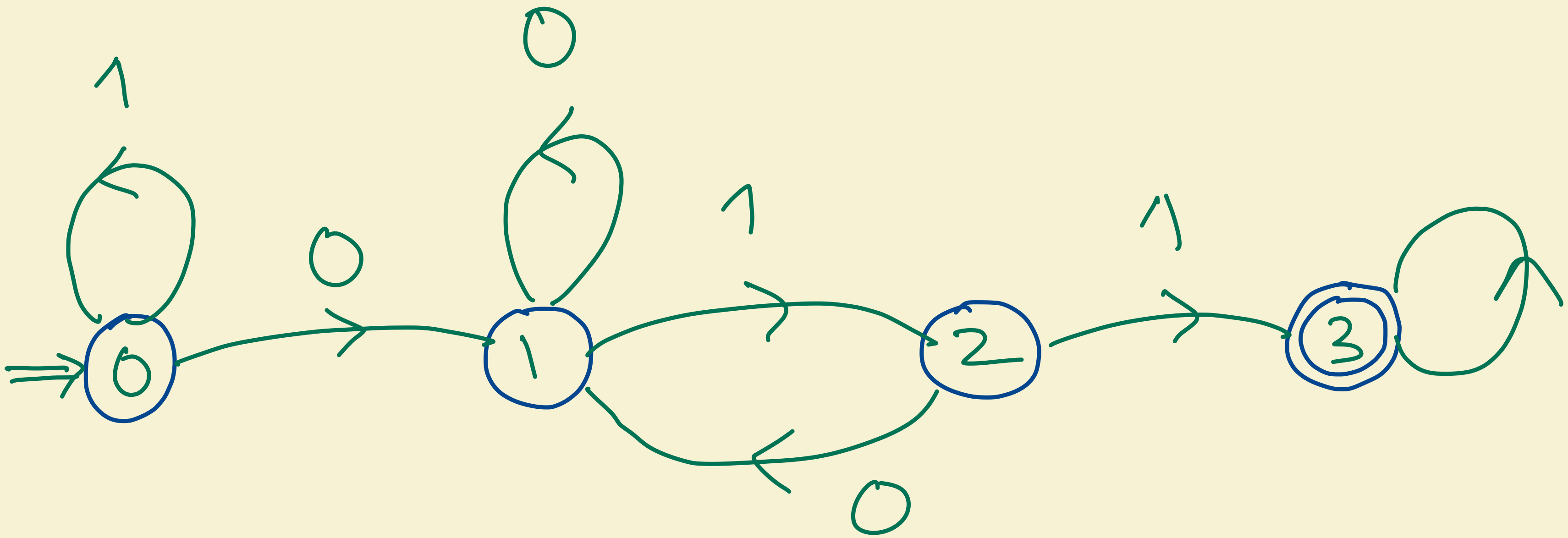
$S = ?$

$f(x) = 1 \Leftrightarrow x$ contains 011 as a subsequence

$S \subseteq \{0, 1, 2, 3\}$

$S = \{3\}$

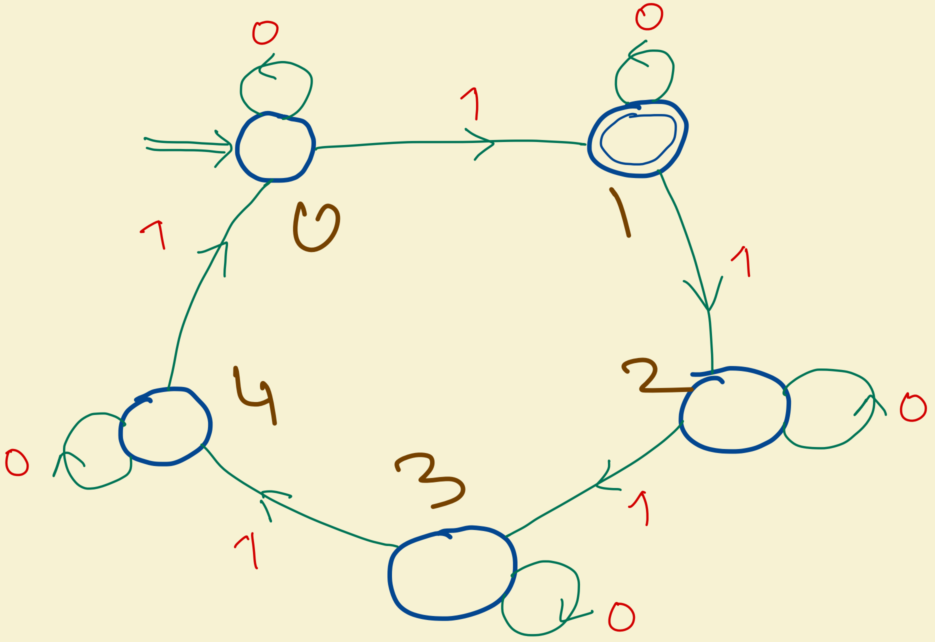
$T(3,1) = 3$



$q \downarrow$ / $\sigma \rightarrow$	0	1
0	1	0
1	1	2
2	1	3
3	3	3

Exercise Break 2:

1) Convert the following diagram to transition function:



$q \backslash \sigma$	0	1
0	0	1
1	1	2
2	2	3
3	3	4
4	4	0

T

2) Describe the function f computed by this DFA.

$f(x) = 1$ if $\sum x_i = 1 \pmod 5$

Regular Expressions

- Motivation: DFA detects simple patterns in strings. Can it do more complex ones?
- Regular expressions:
  - A generalization of "Patterns".
  - Succinct descriptions of subsets of $\{0,1\}^*$
- Definition:
  - Basic cases:
    - 0 is a regular expression
    - 1 is a regular expression
  - Compound cases: If r_1, r_2 are regular expressions, then so are:
    - $r_1 r_2$: "r_1 followed by r_2" (or "concatenation")
    - $(r_1 | r_2)$: "$r_1$or r_2"
    - r_1^*: "Concatenation of finite number of r_1's"
  - End Cases:
    - ϕ (empty set) is regular.
    - "" (null string) is regular.

Regular Expression Matching

~~match~~ regular expression

$(0|1)1(0|1)^*$

1010001

String

- Basic
  - 0 matches 0
  - 1 matches 1
  - “” matches “”
  - No string matches ϕ
- Compound:
  - s matches $r_1 r_2$ if there exists s_1, s_2 such that $s = s_1 s_2$ and s_1 matches r_1 and s_2 matches s_2
  - s matches $(r_1|r_2)$ if s matches r_1 or s matches $\cancel{s_2}$ r_2
  - s matches r_1^* if there exists $s_1, s_2, \dots, s_\ell$ such that $s = s_1 s_2 \dots s_\ell$ and s_i matches r_1 for every $i \in [\ell]$

ℓ could be zero.

Examples:

- $\boxed{(0|1)^*011(0|1)^*}$

  - all strings in $\{0,1\}^*$ match $(0|1)^*$
  - the only string that matches 011 is 011
  - So $(0|1)^* 011 (0|1)^*$ is matched by all strings that have 011 inside as contiguous subsequence.

Examples:

- $(0|1)^*1(0|1)^*1(0|1)^*1(0|1)^*$
  - Again $\{0,1\}^*$ matches $(0|1)^*$
  - so a string must have ≥ 3 1s to match the above.

Examples:

- $(0^*10^*10^*1)^*$
  - $(0^*10^*10^*1)$ is matched by strings with 3 1s, with last character being 1.
  - $(0^*10^*10^*1)^*$ is matched by null string " " & by strings where # of 1's is a multiple of three and last character is a 1.

Regular expressions = sets (languages) = functions

- Can think of a regular expression as a set or as a Boolean function:
  - Given regular expression r can look at set (language)
    - $L(r) = \{x \in \{0,1\}^* \mid x \text{ matches } r\}$
    - $f_r: \{0,1\}^* \to \{0,1\}$ where $f_r(x) = 1 \Leftrightarrow x \in L(r) \Leftrightarrow x \text{ matches } r$

- We prefer the last version

Next two lectures:

- Understanding DFA via regular expressions:
  - For which regular expressions r is $f(r)$ computable by a DFA
    - (Note: # states can depend on r, but not on x or $|x|$)
  - What are some functions computable by DFA that are not regular
- Limits of DFA
  - What are some functions that are not computed by DFA?