CS 121: Lecture 16
Rice’s Theorem

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https://madhu.seas.Harvard.edu/courses/Fall2020

Book: https://introtcs.org

How to contact us

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If eligible, Get Ready to Vote

If you’re voting by mail, make sure to submit your absentee ballot or take action if your ballot has not arrived. Learn more at bit.ly/harvardvotebymail.

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If you have voting questions or need help, text “@votinghelp” to 81010 or email voteschallenge@harvard.edu
Where we are:

Part I: Circuits: 
Finite computation, quantitative study

Part II: Automata: 
Infinite restricted computation, quantitative study

Part III: Turing Machines: 
Infinite computation, qualitative study

Part IV: Efficient Computation: 
Infinite computation, quantitative study

Part V: Randomized computation: 
Extending studies to non-classical algorithms
Review of last two lectures

• \( \text{Cantor}(M) = \overline{M(M)} \) uncomputable
• \( \text{HALT}(M,x) \) uncomputable
• \( E(M) = 1 \iff \forall x, \ M(x) = 0 \) or \( M \) does not halt on \( x \): uncomputable
This lecture

• Uncomputability much more pervasive
  • We’ll keep doing examples until most of the class votes “Go Faster”.
• “Intent of a program” uncomputable
Thm 1: $\text{COMPUTESXOR}(M) = 1$ iff $M(x) = \text{XOR}(x)$ for all $x$. Then $\text{COMPUTESXOR}$ is uncomputable.

Thm 2: $\text{ONLYODD}(M) = 1$ iff $|M(x)|$ odd for all $x$. Then $\text{ONLYODD}$ is uncomputable.

Thm 3: $\text{NOEVENs}(M) = 1$ unless $|M(x)|$ even for some $x$. Then $\text{NOEVENs}$ is uncomputable.

Thm 4: $\text{HALTONSHORT}(M) = 1$ iff $M(x)$ halts whenever $|x| \leq 100$. Then $\text{HALTONSHORT}$ is uncomputable.

Thm 5: $\text{MONOTONE}(M) = 1$ iff $|M(x) \leq M(x')|$ whenever $x \preceq x'$. Then $\text{MONOTONE}$ is uncomputable.

Thm 6: $\text{COMPUTESPAL}(M) = 1$ iff $M(x) = \text{PAL}(x)$ for all $x$. Then $\text{COMPUTESPAL}$ is uncomputable.
Thm 1: $\text{COMPUTESXOR}(M) = 1$ iff $M(x) = \text{XOR}(x)$ for all $x$. Then $\text{COMPUTE\textsc{exor}}$ is uncomputable.

- Recall: $\text{HALT}$ is uncomputable.
- Will use this to prove $\text{COMPUTESXOR}$ is uncomputable.
- I.e. if we could solve $\text{COMPUTESXOR}$, we could solve $\text{HALT}$.
- I.e. we’ll reduce from $\text{HALT}$ to $\text{COMPUTE\textsc{exor}}$
- I.e. we’ll rule out the possibility ($\text{HALT}$ hard, $\text{COMPUTE\textsc{exor}}$ easy)
- I.e. we’ll show that $\text{HALT} \leq \text{COMPUTE\textsc{exor}}$.

**Alg-HALT(x):**
Blah Blah Blah

$z = \text{Alg-}\text{COMPUTE\textsc{exor}}(y)$
Blah blah blah
"COMPUTESXOR uncomputable" doesn’t say:

- ...doesn’t say that there’s no machine that computes XOR.
- i.e. doesn’t say that XOR is uncomputable.
- E.g. Alg-XOR:

```
Alg-XOR(x):
ans = 0
for bit in x:
    ans = ans XOR bit
return ans
```
Proof of Thm 1 (COMPUTESXOR uncomp.)

- **HALT \leq COMPUTEXOR**
- Suppose there exists an algorithm \( ALG – COMPUTEXOR \)...

Alg-HALT(\( M, x \)):
Define \( M_x \) as follows:

- \( M_x(y) \): Simulate \( M \) on \( x \).
  - Ignore the result.
  - Return ALG-XOR(y).

\( z = \text{Alg-COMPUTEXOR}(M_x) \)

Return \( z \)

- This would be an algorithm computing HALT, which doesn’t exist!
Break: discuss proof: COMPUTESXOR uncomp.

- HALT \leq COMPUTEXOR
- Suppose there exists an algorithm ALG – COMPUTEXOR...

Alg-HALT(M, x):
Define \( M_x \) as follows:

\[ M_x(y) : \text{Simulate } M \text{ on } x. \]
\[ \text{Ignore the result.} \]
\[ \text{Return } \text{ALG}-\text{XOR}(y). \]

\[ z = \text{Alg-COMPUTEXOR}(M_x) \]
\[ \text{Return } z \]

- This would be an algorithm computing HALT, which doesn’t exist!
Thm 2: $\text{ONLYODD}(M) = 1$ iff $|M(x)|$ odd for all $x$. Then $\text{ONLYODD}$ is uncomputable.

• Recall: $\text{HALT}$ is uncomputable.
• Will use this to prove $\text{ONLYODD}$ is uncomputable.
• I.e. if we could solve $\text{ONLYODD}$, we could solve $\text{HALT}$.
• I.e. we’ll reduce from $\text{HALT}$ to $\text{ONLYODD}$.
• I.e. we’ll rule out the possibility ($\text{HALT}$ hard, $\text{ONLYODD}$ easy)
• I.e. we’ll show that $\text{HALT} \leq \text{ONLYODD}$.

Alg-$\text{HALT}(x)$:
Blah Blah Blah
$z = \text{Alg-ONLYODD}(y)$
Blah blah blah
“\text{ONLYODD} \ \text{uncomputable}” \ \text{doesn’t say}:

- ...doesn’t say that there’s no machine that outputs odd-length strings on every input.
- i.e. doesn’t say that if a function has only odd-length output, it’s uncomputable.
- E.g. Alg-ODD:

\begin{verbatim}
Alg-ODD(x):
  ignore x.
  return 1001111
\end{verbatim}
Proof of Thm 2 (ONLYODD uncomp.)

- \( \text{HALT} \leq \text{ONLYODD} \)
- Suppose there exists an algorithm \( \text{ALG} \) – \( \text{ONLYODD} \)...

\[
\text{Alg-HALT}(M, x):
\text{Define } M_x \text{ as follows:}
\]

\[
M_x(y): \text{Simulate } M \text{ on } x. \\
\text{Ignore the result.} \\
\text{Return Alg-ODD}(y).
\]

\[
z = \text{Alg-ONLYODD}(M_x) \\
\text{Return } z
\]

- This would be an algorithm computing \( \text{HALT} \), which doesn’t exist!
Recall: \(\text{HALT}\) is uncomputable.

Will use this to prove \(\text{NOEVENS}\) is uncomputable.

I.e. if we could solve \(\text{NOEVENS}\), we could solve \(\text{HALT}\).

I.e. we’ll reduce from \(\text{HALT}\) to \(\text{NOEVENS}\).

I.e. we’ll rule out the possibility (\(\text{HALT}\) hard, \(\text{NOEVENS}\) easy)

I.e. we’ll show that \(\text{HALT} \leq \text{NOEVENS}\).

**Thm 3**: \(\text{NOEVENS}(M) = 1\) unless\(s\) \(|M(x)|\) even for some \(x\). Then \(\text{NOEVENS}\) is uncomputable.
"NOEvens uncomputable" doesn’t say:

- ...doesn’t say that there’s no machine that outputs only odd-length strings or doesn’t halt.
- i.e. doesn’t say that if a function has no even-length output, it’s uncomputable.
- E.g. Alg-LOOP:

```python
Alg-LOOP(x):
   while(True)
```

- ...doesn’t say the opposite, either!
- i.e. doesn’t say that if a function has even-length output, it’s uncomputable.
- E.g. Alg-EVEN:

```python
Alg-EVEN(x):
   return "121_is_great"
```
Proof of Thm 3 (NOEVENS uncomp.)

- \( \text{HALT} \leq \text{NOEVENS} \)
- Suppose there exists an algorithm \( \text{ALG} - \text{NOEVENS} \)...

\[
\text{Alg-HALT}(M, x): \\
\text{Define } M_x \text{ as follows:} \\
\text{\( M_x(y) \): Simulate } M \text{ on } x. \\
\text{Ignore the result.} \\
\text{Return } 1-(\text{Alg-EVEN}(y)).
\]

\[ z = \text{Alg-NOEVENS}(M_x) \]

\( \text{Return } z \)

- This would be an algorithm computing HALT, which doesn’t exist!
Recall: $\text{HALT}$ is uncomputable.

Will use this to prove $\text{HALTONSHORT}$ is uncomputable.

I.e. if we could solve $\text{HALTONSHORT}$, we could solve $\text{HALT}$.

I.e. we’ll reduce from $\text{HALT}$ to $\text{HALTONSHORT}$.

I.e. we’ll rule out the possibility ($\text{HALT}$ hard, $\text{HALTONSHORT}$ easy)

I.e. we’ll show that $\text{HALT} \leq \text{HALTONSHORT}$.

**Thm 4:** $\text{HALTONSHORT}(M) = 1$ iff $M(x)$ halts whenever $|x| \leq 100$.

Then $\text{HALTONSHORT}$ is uncomputable.

**Alg-HALT($x$):**
Blah Blah Blah

$z = \text{Alg-HALTONSHORT}(y)$
Blah blah blah
"HALTONSHORT uncomputable" doesn’t say:

• ...doesn’t say that there’s no machine that halts on short inputs.
• E.g. Alg-EVEN:

  Alg-EVEN(x):
  return ``121_is_great``

• ...doesn’t say the opposite, either!
• E.g. Alg-LOOP:

  Alg-LOOP(x):
  while(True)
Proof of Thm 4 (HALTONSHORT uncomp.)

- \( \text{HALT} \leq \text{HALTONSHORT} \)
- Suppose there exists an algorithm \( \text{ALG} - \text{HALTONSHORT} \)...

\[ \text{Alg-HALT}(M, x): \]
Define \( M_x \) as follows:

\[ M_x(y): \text{Simulate } M \text{ on } x. \]
Ignore the result.
Return Alg-EVEN(y).

\[ z = \text{Alg-HALTONSHORT}(M_x) \]
Return \( z \)

- This would be an algorithm computing \( \text{HALT} \), which doesn’t exist!
Rice’s Thm: For every $F: \{0,1\}^* \rightarrow \{0,1\}$, if $F$ is semantic then either $F = \text{one}$ or $F = \text{zero}$ or $F$ is uncomputable.

Def: $M$ and $M'$ are functionally equivalent if for every $x \in \{0,1\}^*$, $M(x) = M'(x)$
Notation: $M \equiv M'$

Def: $F: \{0,1\}^* \rightarrow \{0,1\}$ is semantic if for every $M \equiv M'$, $F(M) = F(M')$

Q: Let $\text{one:} \{0,1\}^* \rightarrow \{0,1\}$ be constant one function ($\text{one}(w) = 1$ for every $w$). Then $\text{one}$ is semantic.
Rice’s Theorem: If $F: \{\text{Turing Machines} \} \rightarrow \{0,1\}$ has property that $F(M)$ is *semantic* (only depends on what $M$ computes, not how) and $F$ is not *trivial* (true for every $M$ or no $M$), then $F$ is uncomputable.
Thm n (Rice’s Theorem)

• Recall: \textit{HALT} is uncomputable.
• Will use this to prove \textit{F} is uncomputable.
• I.e. if we could solve \textit{F}, we could solve \textit{HALT}.
• I.e. we’ll reduce from \textit{HALT} to \textit{F}.
• I.e. we’ll rule out the possibility (\textit{HALT} hard, \textit{F} easy)
• I.e. we’ll show that \textit{HALT} \leq \textit{F}.

\begin{itemize}
  \item Alg-\textit{HALT}(x): \\
  \text{Blah Blah Blah} \\
  z = \text{Alg-}F(y) \\
  \text{Blah blah blah}
\end{itemize}
“F isn’t trivial” means:

- There’s some machine M such that $F(M) = 1$.
- E.g. Alg-???:

  \[
  \text{Alg-at-least-it-exists}(x): \quad ???
  \]

- There’s some machine M such that $F(M) = 0$
- E.g. Alg-LOOP:

  \[
  \text{Alg-LOOP}(x): \quad \text{while(True)}
  \]

(These might be switched.)
Proof of Thm n (Rice’s theorem: F uncomp.)

- $\text{HALT} \leq F$
- Suppose there exists an algorithm $\text{ALG} - F$ ...

Alg-HALT($M, x$):
Define $M_x$ as follows:

$M_x(y)$: Simulate $M$ on $x$.
Ignore the result.
Return Alg-at-least-it-exists($y$).

$z = \text{Alg-F}(M_x)$
Return $z$

- This would be an algorithm computing $\text{HALT}$, which doesn’t exist!
Break: discuss Rice’s Theorem & proof

• HALT ≤ F
• Suppose there exists an algorithm \textit{ALG} − F...

Alg-HALT(M, x):
Define $M_x$ as follows:

$M_x(y)$: Simulate $M$ on $x$.
Ignore the result.
Return Alg-at-least-it-exists(y).

$z = \text{Alg-F}(M_x)$
Return $z$

• This would be an algorithm computing HALT, which doesn’t exist!
Rice’s Theorem caveats

Rice’s Thm: For every $F: \{0,1\}^* \rightarrow \{0,1\}$, if $F$ is semantic then either $F = \text{one}$ or $F = \text{zero}$ or $F$ is uncomputable.

A first approximation is “functions that take a TM $M$ as input are uncomputable”, but:

- Check whether you actually needed the TM as input.

- Functions that ask about how $M$ computes things might or might not be computable. “$M$ has <100 states” vs “$M$ has <100 states and computes HALT”

...this isn’t just a loophole; it lets us salvage things we want from uncomputability!
Things we’d like to do, but can’t.

**Bug checking:** Given a program M (say, in Python), check whether it will always return.

**Type checking:** Given a program M (say, in Python), check whether it ever calls the `concat` function with inputs that aren’t strings.

**Equivalence checking:** A clever programmer claims to have found a faster replacement for your function. It’s fast, but you don’t understand the code. Is it right?
Rice’s Theorem (Fundamental Theorem of Software Verification):
Every semantic $F$ is either trivial or uncomputable.

Is software verification doomed?

Q: Let $ValidType : \{0,1\}^* \rightarrow \{0,1\}$ be function that maps a C program $P$ to 1 iff when $P$ is executed, it will never call a function with a char parameter with an int parameter.
Prove that $ValidType$ is computable.

Type mismatch error
Coping with Rice

- Turing Machines
- General programming languages
- \(\lambda\) calculus
- ...

Restricted Computational Models
- Weaker
  + Semantic analysis

Complete Models
- Stronger
  - No semantic analysis

Type systems
- Proof systems
  - Simply typed \(\lambda\) calculus
  - System F
  - HTML
  - Context Free Grammars
  - Regular expressions
  - ANSI SQL (92)
  - ...

Turing Machines
- General programming languages
- \(\lambda\) calculus
- ...

Simply typed \(\lambda\) calculus
- System F
- HTML
- Context Free Grammars
- Regular expressions
- ANSI SQL (92)
- ...

Restricted Computational Models
- Weaker
  + Semantic analysis
Section + Next Lecture

• Section: More Uncomputability + Reductions
  • HALT-ON-ZERO
    • H-O-Z(M) = 1 if M accepts "" and 0 otherwise.
  • Moral: It is not the infinity of inputs that makes HALT hard!

• Next Lecture: Efficient Computation (P)