CS 121: Lecture 21
More NP-completeness by Reductions

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https://madhu.seas.Harvard.edu/courses/Fall2020

Book: https://introtcs.org

How to contact us

- The whole staff (faster response): CS 121 Piazza
- Only the course heads (slower): cs121.fall2020.course.heads@gmail.com
Announcements:

• 121.5: Nicole Immorlica: Econ and CS
• Sections: Polynomial time reductions, NP, etc.
• Homework 5 due today.
• Midterm 2 this Tuesday!
  • 90 minutes (70 if handwritten)
  • 2-sided cheatsheet, noncollaboratively made, plus Barak’s textbook.
  • Material through lecture 17 (Efficient Computation: P)
Where we are:

Part I: Circuits:
Finite computation, quantitative study

Part II: Automata:
Infinite restricted computation, quantitative study

Part III: Turing Machines:
Infinite computation, qualitative study

Part IV: Efficient Computation:
Infinite computation, quantitative study

Part V: Randomized computation:
Extending studies to non-classical algorithms
Review of last lectures

- Reductions: $F \leq_P G \iff \exists R \text{ such that } \forall x \ F(x) = G(R(x)), \ R \text{ polytime.}$
- $3\text{SAT} \leq_P \text{ISET}$

- NP: problems easy to verify.

  $F: \{0,1\}^* \to \{0,1\}$ is in NP iff:

  \[ \exists V_F: \{0,1\}^* \times \{0,1\}^* \to \{0,1\} \text{ s.t. } \forall x \in \{0,1\}^*, \ \\
  F(x) = 1 \iff \exists w \in \{0,1\}^* \text{ such that } V_F(x,w) = 1 \ \\
  \text{and } V_F(x,w) \text{ computable in time } \text{poly}(|x|) \]

- (Any problem in NP) $\leq_P \text{NANDSAT} \leq_P 3\text{NAND} \leq_P 3\text{SAT}$

  - So 3SAT is NP-Complete!
Witness, the NP concept

Function $F$ is in NP if $\exists$ polytime $V_F$ s.t. $(F(x) = 1) \iff (\exists w: V_F(x, w) = 1)$

<table>
<thead>
<tr>
<th>Function $F$</th>
<th>Witness $w$</th>
<th>Verifier $V_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3SAT(formula)</td>
<td>Variable values</td>
<td>Check: formula satisfied?</td>
</tr>
<tr>
<td>Longpath(G)</td>
<td>Sequence of vertices</td>
<td>Check: is path, is long</td>
</tr>
<tr>
<td>COMPOSITE(x)</td>
<td>Factors $p, q$</td>
<td>Check: $p*q=x$</td>
</tr>
<tr>
<td>COMPOSITE(x)</td>
<td>$y, z$</td>
<td>Check: $\frac{yz}{x} \in \mathbb{Z}, \frac{y}{x} \notin \mathbb{Z}, \frac{z}{x} \notin \mathbb{Z}$</td>
</tr>
</tbody>
</table>
Witness, the computer game

Figure 1: A screenshot from The Witness, featuring 2D puzzles in a 3D world.
Today:

• Some NP-complete problems...
• $3\text{SAT} \leq_p \text{E3SAT} \leq_p \text{EU3SAT} \leq_p 1\text{-in-EU3SAT} \leq_p \text{SUBSETSUM}$
• Weak NP-hardness: hard only for big-number inputs
• Strong NP-hardness: hard even for small-number inputs.
3SAT $\leq_P$ E3SAT

Last time, 3NAND $\leq_P$ 3SAT:

$$c = \text{NAND}(a, b)$$

3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29})$,
\textit{at most} 3 variables/clause

E3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29} \lor x_{22})$,
\textit{exactly} 3 variables/clause.
3SAT $\leq_p$ E3SAT

**Reduction:**

\[
(x_7) \quad \Rightarrow \quad (x_7 \lor x_7 \lor x_7)
\]

\[
(x_7 \lor \overline{x}_{17}) \quad \Rightarrow \quad (x_7 \lor \overline{x}_{17} \lor x_7)
\]

\[
(x_7 \lor \overline{x}_{17} \lor x_{29}) \quad \Rightarrow \quad (x_7 \lor \overline{x}_{17} \lor x_{29})
\]

\[
(x_7) \land (x_7 \lor \overline{x}_{17}) \land (x_7 \lor \overline{x}_{17} \lor x_{29}) \quad \Rightarrow \quad (x_7 \lor x_7 \lor x_7) \land (x_7 \lor \overline{x}_{17} \lor x_7) \land (x_7 \lor \overline{x}_{17} \lor x_{29})
\]

**Proof:**

(Sound, Complete)

\[
(x_7) \land (x_7 \lor \overline{x}_{17}) \land (x_7 \lor \overline{x}_{17} \lor x_{29})
\]

is satisfiable

\[
(x_7 \lor x_7 \lor x_7) \land (x_7 \lor \overline{x}_{17} \lor x_7) \land (x_7 \lor \overline{x}_{17} \lor x_{29})
\]

is satisfiable

\[
(x_7 \lor \overline{x}_{17})
\]

is satisfiable

\[
(x_7 \lor \overline{x}_{17} \lor x_7)
\]

is satisfiable

with the same variable values
3SAT $\leq_P$ E3SAT

Reduction:

Proof:

(Sound, Complete)

Q: Have we proved that E3SAT is NP-complete?
$E3SAT \leq_P EU3SAT$

3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29})$, 

*at most* 3 variables/clause

E3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29} \lor x_{22})$, 

*Exactly* 3 variables/clause.

EU3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29} \lor x_{23})$, 

*exactly* 3 *unique* variables/clause.
\[ E3SAT \leq_P EU3SAT \]

**Reduction:**

\[ (x_7 \lor \overline{x}_{17} \lor x_7) \rightarrow (x_7 \lor \overline{x}_{17} \lor y_7) \land (x_7 \lor \overline{y}_7 \lor temp) \land (x_7 \lor \overline{y}_7 \lor \overline{temp}) \land (\overline{x}_7 \lor y_7 \lor temp) \land (\overline{x}_7 \lor y_7 \lor \overline{temp}) \]

(Wherever we have \( t \) copies of a variable in a clause, change \( t-1 \) of them and add \( 4(t-1) \) clauses.)

**Proof:**

(Sound, Complete)

E3SAT formula with clauses like 
\[ (x_7 \lor \overline{x}_{17} \lor x_7) \] is satisfiable

EU3SAT formula with clauses like 
\[ (x_7 \lor \overline{x}_{17} \lor y_7) \land (x_7 \lor \overline{y}_7 \lor temp) \land (x_7 \lor \overline{y}_7 \lor \overline{temp}) \land (\overline{x}_7 \lor y_7 \lor temp) \land (\overline{x}_7 \lor y_7 \lor \overline{temp}) \] is satisfiable
$\textbf{EU3SAT} \leq_P \textbf{1-in-EU3SAT}$

3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29})$, 
\emph{at most} 3 variables/clause, clause is satisfied iff \textbf{at least} one literal is true.

E3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29} \lor x_{22})$, 
\emph{Exactly} 3 variables/clause, clause is satisfied iff \textbf{at least} one literal is true.

EU3SAT: Formulas like $(x_7 \lor \overline{x}_{17} \lor x_{29}) \land (\overline{x}_7 \lor x_{15} \lor x_{22}) \land (x_{22} \lor \overline{x}_{29} \lor x_{23})$, 
\emph{exactly} 3 \emph{unique} variables/clause, clause is satisfied iff \textbf{at least} one literal is true.

1-in-EU3SAT: Formulas like $\text{ONEOF}(x_7, \overline{x}_{17}, x_{29}) \land \text{ONEOF}(\overline{x}_7, x_{15}, x_{22}) \land \text{ONEOF}(x_{22}, \overline{x}_{29}, x_{23})$, 
\emph{exactly} 3 \emph{unique} variables/clause, clause is satisfied iff \textbf{exactly} one literal is true.
**EU3SAT \( \leq_p 1\text{-in-EU3SAT} \)**

**Reduction:**

\[(a \lor b \lor c) \rightarrow ONEOF(\overline{a}, w, x) \land ONEOF(b, y, x) \land ONEOF(c, w, z)\]

**Proof:**

(Sound, Complete)
More SAT variants...

Figure 2-1: SAT notation example.
Knapsack Problem:

Given items with costs $a_0, a_1, ..., a_{k-1}$ and values $v_0, v_1, ..., v_{k-1}$, a budget $b$, and a target value $t$, choose a subset of the items with total cost at most $b$ and value at least $t$. 
Knapsack Problem:

Given items with costs $a_0, a_1, \ldots, a_{k-1}$ and values $v_0, v_1, \ldots, v_{k-1}$, a budget $b$, and a target value $t$, choose a subset of the items with total cost at most $b$ and value at least $t$.

Subset Sum:

Given items with costs $a_0, a_1, \ldots, a_{k-1}$ and a target value $t$, choose a subset of the items with total cost exactly $t$. 
1-in-EU3SAT $\leq_P$ Subset Sum

Formulas like
ONEOF($x_7, \overline{x}_{17}, x_{29}$) \land
ONEOF($\overline{x}_7, x_{15}, x_{22}$) \land
ONEOF($x_{22}, \overline{x}_{29}, x_{23}$)

Subset Sum numbers (written in base $n + 1$)

\[\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 \ a_0 \\
0 & 1 & 0 & 0 & 0 & 1 \ a_1 \\
0 & 0 & 0 & 0 & 1 & 0 \ a_2 \\
0 & 0 & 1 & 0 & 1 & 0 \ a_3 \\
1 & 0 & 0 & 1 & 0 & 0 \ a_{2n-2} \\
0 & 0 & 0 & 1 & 0 & 0 \ a_{2n-1} \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & t
\end{array}\]

Given items with costs $a_0, a_1, ..., a_{k-1}$ and a target value $t$, choose a subset of the items with total cost exactly $t$.

Reduction:

1-in-EU3SAT formula
m clauses (here $m=3$)
n variables (here $n=7$)

ONEOF($x_7, \overline{x}_{17}, x_{29}$) \land ONEOF($\overline{x}_7, x_{15}, x_{22}$) \land ONEOF($x_{22}, \overline{x}_{29}, x_{23}$)

Proof of Correctness?
Weak NP-hardness

Subset sum: Given items with costs $a_0, a_1, \ldots, a_{k-1}$ and a target value $t$, choose a subset of the items with total cost exactly $t$.

Some numbers (costs) in reduction were exponential in $n$. (Poly length!)
If all inputs were polynomial in $n$, Subset Sum isn’t NP-hard.

“Weakly NP-hard”

“Strongly NP-hard”: NP-hard even if all numerical inputs are polynomial-sized.
Traveling Salesman:

Given a (directed or undirected) graph $G$, a “distance” $d_e$ for each edge $e$, and a target $t$, is there a walk visiting all the vertices of $G$ whose total distance is at most $t$?

Strongly NP-hard (NP-hard even if $t$ and every $d_e$ is small).

Hint: Reduce from Longpath: Given a (directed or undirected) graph $G$ and a target $t$, is there a path visiting at least $t$ vertices? (Paths can’t revisit vertices.)
Summary of Lecture:

• \(3\text{SAT} \leq_p \text{E3SAT} \leq_p \text{EU3SAT} \leq_p \text{1-in-EU3SAT} \leq_p \text{SUBSETSUM}\)
• Weak NP-hardness: hard only for big-number inputs
• Strong NP-hardness: hard even for small-number inputs.