CS 121: Lecture 24
Intro to Randomized Algorithms

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https://madhu.seas.Harvard.edu/courses/Fall2020

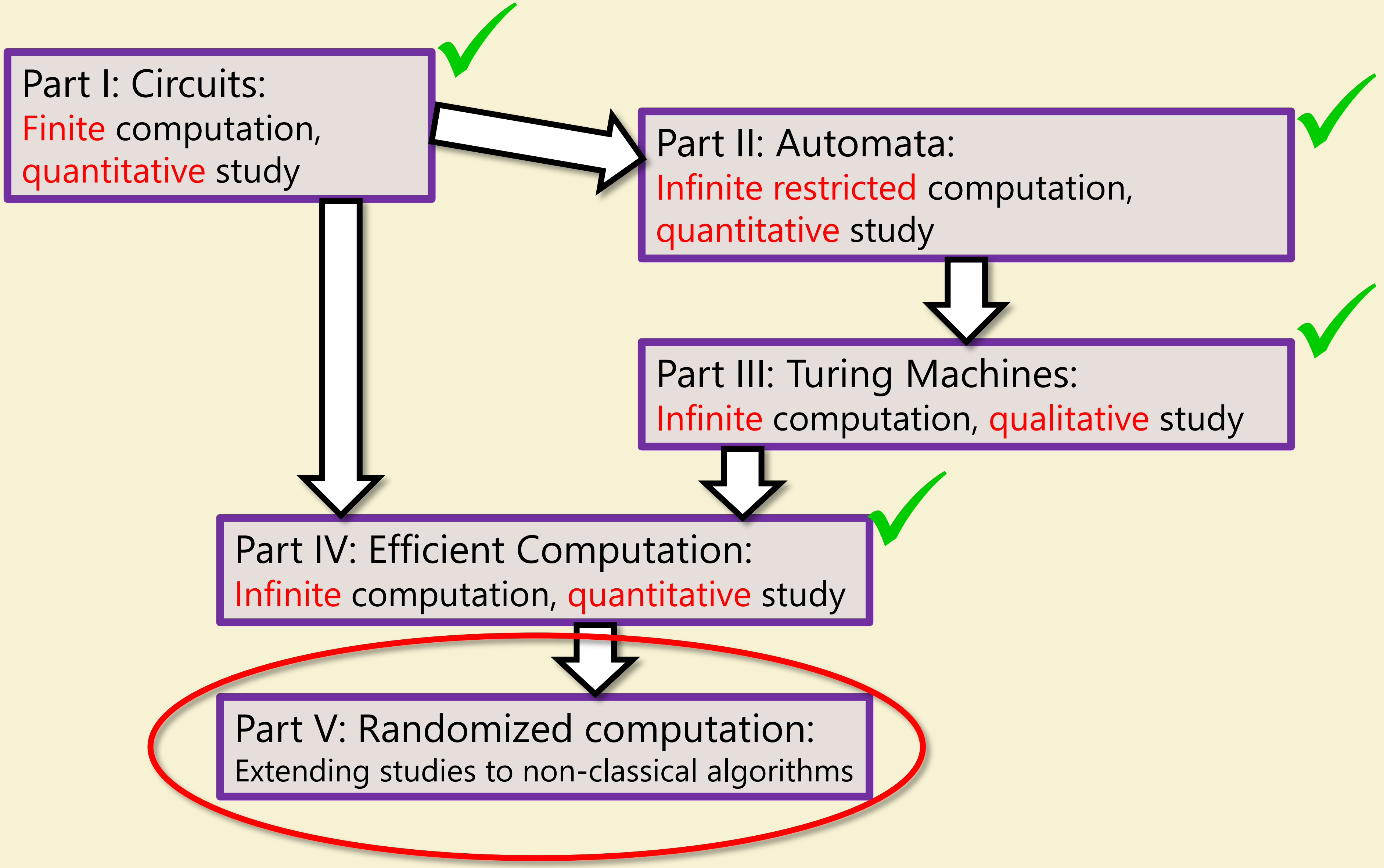
Book: https://introtcs.org

How to contact us
- The whole staff (faster response): CS 121 Piazza
- Only the course heads (slower): cs121.fall2020.course.heads@gmail.com

Announcements:

- Midterm 2 graded. Solutions to be posted today-ish.
- Thanks for participating in Midterm Feedback Survey.
- Happy Thanksgiving! (Next lecture Tuesday.)

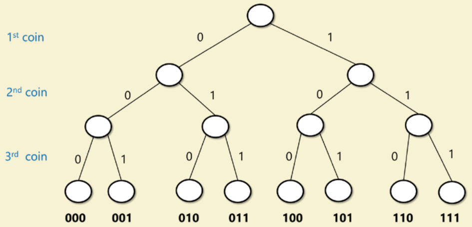
Where we are:

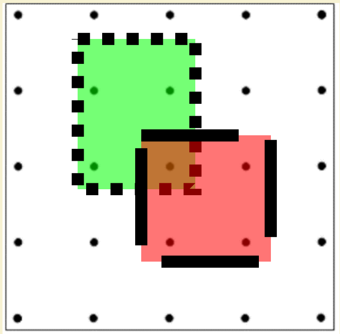


Part I: Circuits:
Finite computation, quantitative study
Part II: Automata:
Infinite restricted computation, quantitative study
Part III: Turing Machines:
Infinite computation, qualitative study
Part IV: Efficient Computation:
Infinite computation, quantitative study
Part V: Randomized computation:
Extending studies to non-classical algorithms

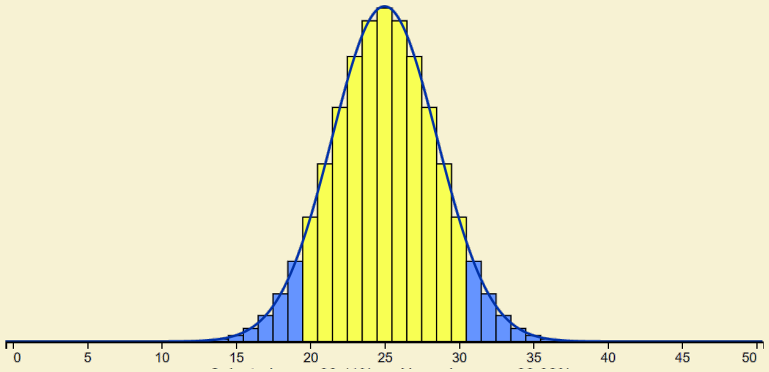
Last lecture

- Sample space





- Events
- Union/intersection/negation – AND/OR/NOT of events
- Random variables $X: \{0,1\}^n \to \mathbb{R}$
- Expectation Average value of X : $\mathbb{E}[X] = \sum_{x \in \{0,1\}^n} 2^{-n} X(x) = \sum_{v \in \mathbb{R}} v \cdot \Pr[X = v]$
- Concentration / tail bounds



Today:

- Randomized Algorithms
  - Polynomial Identity Testing
  - Approximation for maximum cut
- Randomized Complexity Class BPP
- Properties of randomized computation (Reducing error …)

Informal

A randomized algorithm has a special operation:

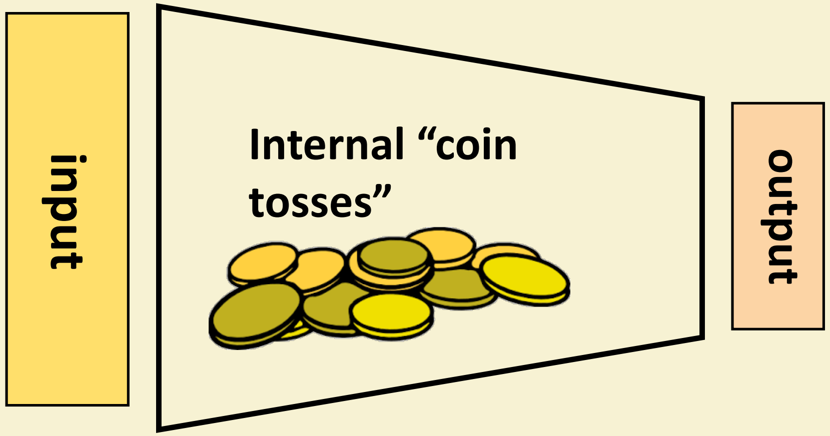
`foo` =

i.e. `foo` $\sim \{0,1\}$

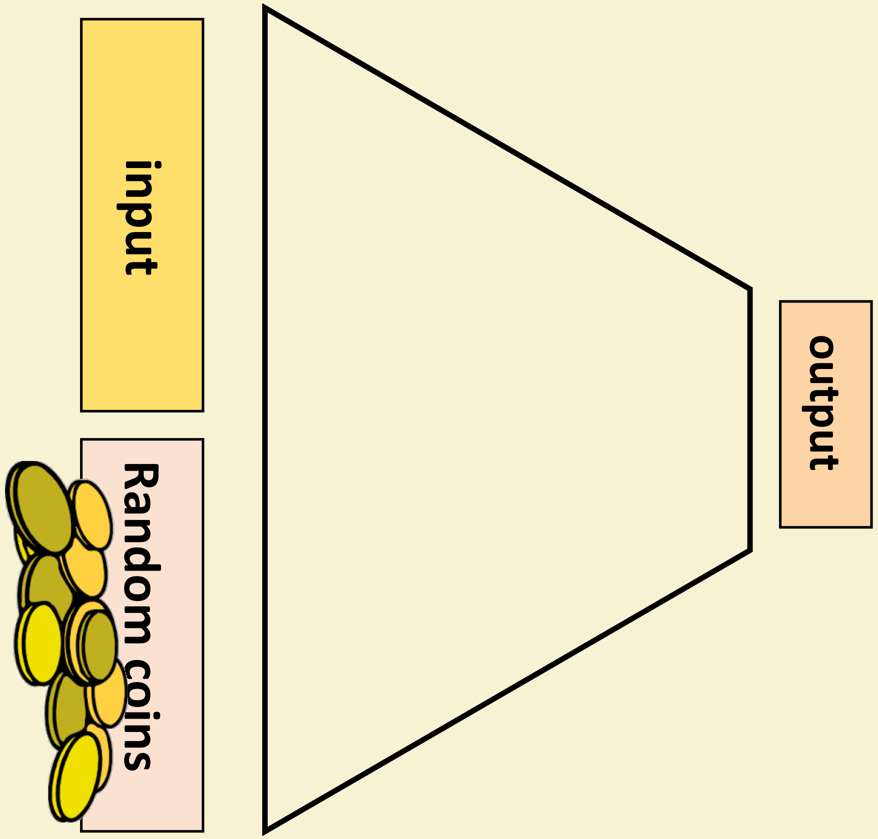
By repeating can choose `foo` $\sim \{0,1\}^n$ or $\sim [0,1]$

Randomized algorithms

Two equivalent views:



1. Get input $x \in \{0,1\}^n$
2. Run alg $A(x)$ that has special operation $r_i \leftarrow RAND()$ $(r_i \sim \{0,1\})$



1. Get input $x \in \{0,1\}^n$
2. Choose $r \sim \{0,1\}^m$
3. Run deterministic algorithm $A(x,r)$

$$output = ALG(input, randomness)$$

Computing a function

Not random input – has to work in the worst case

Randomized algorithm ALG computes F if for every input x

$$\Pr[ALG(x) = F(x)] \geq \frac{2}{3}$$

Probability over the randomness of the algorithm, not the input

The constant $2/3$ is arbitrary – can be replaced by 0.51, 0.99, even $1 - 2^{-n}$. Not by $1/2$.

BPP: {Boolean functions computable by some randomized algorithm}

Polynomial Identity Testing: Problem

Q: $(x + yz)^7 - x^7 - y^7 z^7 = 7x(x + yz)(x^2 + y^2 z^2)(x^2 + xyz + y^2 z^2)$?

Standard form: $(x + yz)(x + yz)(x + yz)(x + yz)(x + yz)(x + yz)(x + yz) - xxxxxxx - yyyyyyyzzzzzzz - 7x(x + yz)(xx + yyzz)(xx + xyz + yyzz) = 0$?

Input φ: an expression like the above, with sums/products of variables.

Output $PIT(\varphi)$: 1 iff φ is the 0 polynomial.

Why is the following not a polynomial-time algorithm for PIT?

Alg-PIT(φ):

Multiply everything out,

Add/subtract like terms,

Return 1 iff all terms cancel.

Polynomial Identity Testing: Algorithm

Q: $(x+yz)(x+yz)(x+yz)(x+yz)(x+yz)(x+yz)(x+yz) - xxxxxxx - yyyyyyyzzzzzzz - 7x(x+yz)(xx+yyzz)(xx+xyz+yyzz) = 0$?

Randomized algorithm for PIT (note: polynomial time!):

RandAlg-PIT(φ):

For each variable, choose a random number between 0 and 3n.

Plug in those values and do all the integer arithmetic.

Return 1 iff the result is 0.

Can give the wrong answer! Give an example.

.

Polynomial Identity Testing: Correctness (1/2)

Randomized algorithm for PIT:

> RandAlg-PIT(φ):
>
> For each variable, choose a random number between 0 and 3n.
>
> Plug in those values and do all the integer arithmetic.
>
> Return 1 iff the result is 0.

Goal: $\Pr[\, RandAlg - PIT(x) = PIT(x)] \geq \frac{2}{3}$

If $PIT(\varphi) = 1$, $\Pr[\, RandAlg - PIT(\varphi) = 1] = 1$

If $PIT(\varphi) = 0$…

Polynomial Identity Testing: Correctness (2/2)

$(x+yz)(x+yz)(x+yz)(x+yz)(x+yz)(x+yz)(x+yz) - xxxxxxx - yyyyyyyzzzzzzz = 7x(x+yz)(xx+yyzz)(xx+xyz+yyzz)$?

RandAlg-PIT(φ):

For each variable, choose a random number between 0 and 3n.

Plug in those values and do all the integer arithmetic.

Return 1 iff the result is 0.

If $PIT(\varphi) = 0$: note that the degree is at most n.

Fact: A 1-variable polynomial $p \neq 0$ is 0 for $\leq \deg(p)$ inputs in $\{0, \dots, 3n\}$

Fact: A k-variable polynomial $p \neq 0$ is 0 for $\leq \deg(p)(3n+1)^{k-1}$ inputs in $\{0, \dots, 3n\}^k$

So $\Pr[\,RandAlg - PIT(\varphi) = 0] = \Pr[\varphi(x) = 0] \leq \frac{\deg(p)}{3n+1} < \frac{2}{3}$.

Success amplification

We have an algorithm RandAlg-PIT for which:

$$\Pr[RandAlg - PIT(\varphi) = F(x)] \geq \frac{2}{3}$$

Give an algorithm BetterRandAlg-PIT for which:

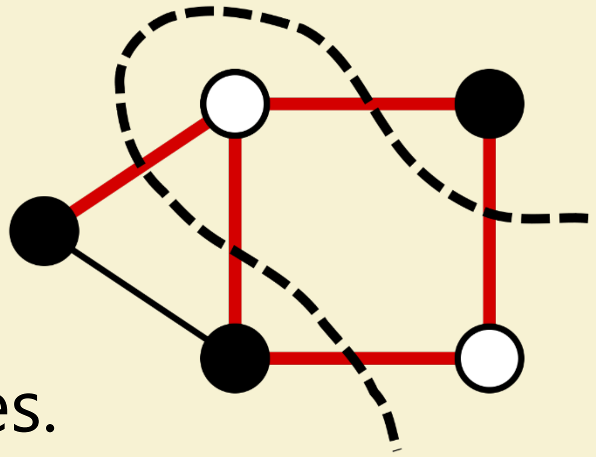
$$\Pr[RandAlg - PIT(\varphi) = F(x)] \geq 1 - 2^{-60}$$

Note: $\Pr[\,failure] < \Pr[\,asteroid\ hits\ us\ this\ minute]$

Bottom line: randomized algorithms as good as deterministic for all practical purposes.

Recall: randomized algorithms – work on worst case inputs.
Randomness is only over the coins of the algorithm.

MAXCUT



Input: Graph $G = (V, E)$.

Output: Partition of V maximizing # of crossing edges.

Define: $OPT(G) = \max_{S \subseteq V} |E(S, \overline{S})|$ to be max # of cut edges.

If $P \neq NP$,no poly-time alg computes $OPT(G)$ / produces cut achieving it.

We'll show: Poly-time randomized algorithm that w/ probability ≥ 0.99 outputs cut S that cuts at least $0.5 \cdot OPT(G)$ edges.

Best known: Alg cutting $\alpha \cdot OPT(G)$ edges for $\alpha = \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \cdot \frac{\theta}{1 - \cos\theta} \approx 0.87857$

Central open question: is this optimal?

Input: Graph $G = (V, E)$.

Output: Partition of V maximizing # of cuts crossing edges.

Define: $OPT(G) = \max_{S \subseteq V} |E(S, \overline{S})|$ to be max # of cut edges.

We'll show: Poly-time randomized algorithm that w/ probability ≥ 0.99 outputs cut S that cuts at least $0.5 \cdot OPT(G)$ edges.

Thm: $\exists$ randomized poly time algorithm A s.t. with prob ≥ 0.99

$$A(G) = S \text{ s.t. } |E(S, \overline{S})| \geq |E|/2$$

Q: Why does Thm imply what we need to show?

Thm: $\exists$ randomized poly time algorithm A s.t. with prob ≥ 0.99

$$A(G) = S \text{ s.t. } \left|E\left(S,\overline{S}\right)\right| \geq |E|/2$$

Lemma: $\exists$ randomized poly time algorithm A s.t. if $S = A(G)$ then

$$\mathbb{E}\left[\left|E\left(S,\overline{S}\right)\right|\right] \geq |E|/2$$

Over randomness of A

Q: Why does Lemma not immediately imply the theorem?

Lemma: $\exists$ randomized poly time algorithm A s.t. if $S = A(G)$ then

$$\mathbb{E}\left[\left|E\left(S,\overline{S}\right)\right|\right] \geq |E|/2$$

Proof: Given G on n vertices, A picks $x \sim \{0,1\}^n$ and output $S = \{\, i \mid x_i = 1 \,\}$

For every edge $(i,j) \in E$, define $X_{i,j} = \begin{cases} 1, & x_i \neq x_j \\ 0, & x_i = x_j \end{cases}$

Q: What is $\mathbb{E}[X_{i,j}]$? A: $1/2$

Q: Prove that $\left|E\left(S,\overline{S}\right)\right| = \sum_{(i,j)\in E} X_{i,j}$

∎

From expectation to high probability

Success amplification

Given: Poly-time alg A s.t. that $\mathbb{E}\left[val(A(G))\right] \geq k$

Goal: Poly-time alg B s.t. that $\Pr\left[val\big(B(G)\big) \geq k\right] \geq 0.99$

Algorithm B

Input: G

m: # of edges

for $i = 1 \ldots 1000m$:

  $S_i \leftarrow A(G)$ # *fresh randomness each time*

return S_i maximizing edges cut

Given: Poly-time alg A s.t. that $\mathbb{E}\left[val(A(G))\right] \geq k$

Goal: Poly-time alg B s.t. that $\Pr[val(B(G)) \geq k] \geq 0.99$

Algorithm B

Input: G

for $i = 1 \ldots 1000m$:

  $S_i \leftarrow A(G)$ *# fresh randomness each time*

return S_i maximizing edges cut

Lemma: $\Pr[val(A(G)) \geq k] \geq 1/m$

Q: Prove that Lemma $\Rightarrow \Pr[val(B(G)) \geq k] \geq 0.99$

Given: Poly-time alg A s.t. that $\mathbb{E}\left[val(A(G))\right] \geq k$

Lemma: $\Pr[val(A(G)) \geq k] \geq 1/m$

Proof: Suppose that $\Pr[\ val(A(G)) \geq k\ ] < \frac{1}{m}$

then

prob $< 1/m$

val $\leq m$

prob ≤ 1

val $\leq k - 1$

$$\mathbb{E}[val(A(G)] \quad < \quad \underbrace{\frac{1}{m} \cdot m}_{} \quad + \quad \underbrace{1 \cdot (k-1)}_{} \quad = k$$

Contribution from case that $val(A(G)) \geq k$

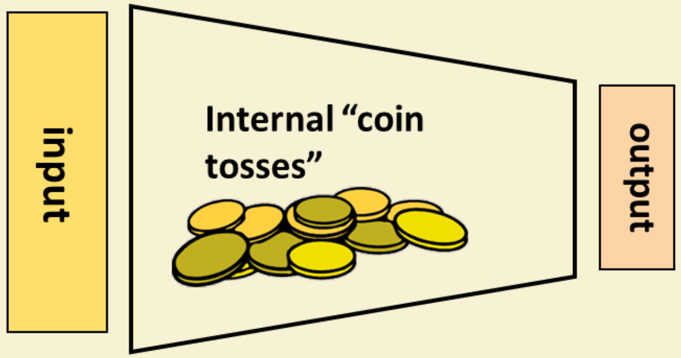
Contribution from case that $val(A(G)) < k - 1$

∎

Recap

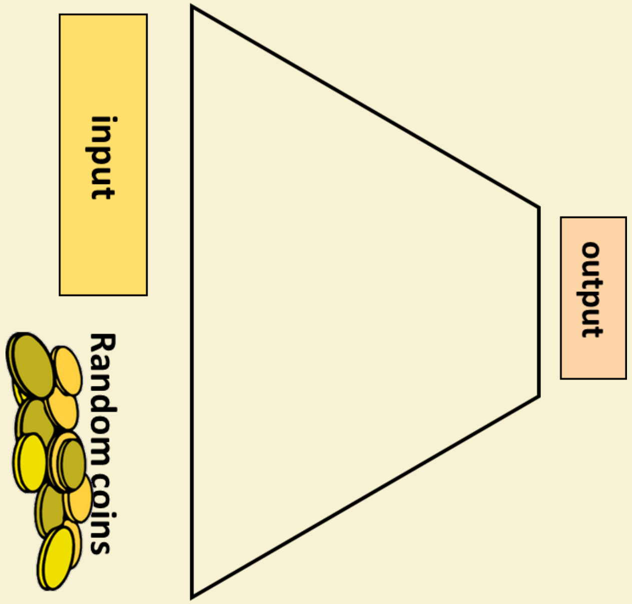
Def: $F: \{0,1\}^* \to \{0,1\}$ is in BPP is there is a poly-time randomized algorithm A s.t. $\forall n\ \forall x \in \{0,1\}^n$

$$\Pr_{A's\ randomness}[A(x) = F(x)] \geq \frac{2}{3}$$

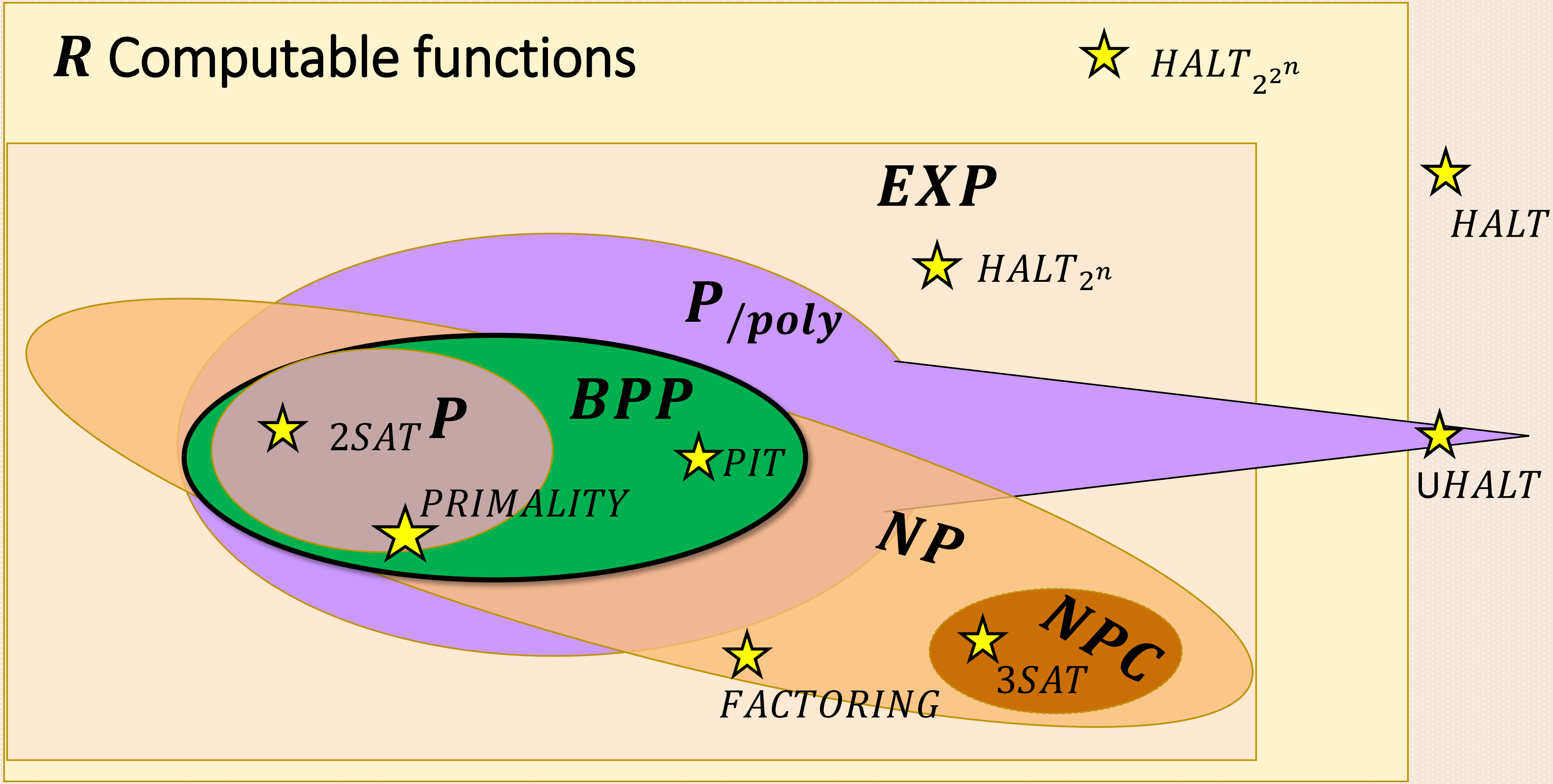


Def: $F: \{0,1\}^* \to \{0,1\}$ is in BPP is there is a poly-time algorithm A , poly $q(n)$ s.t. $\forall n\ \forall x \in \{0,1\}^n$

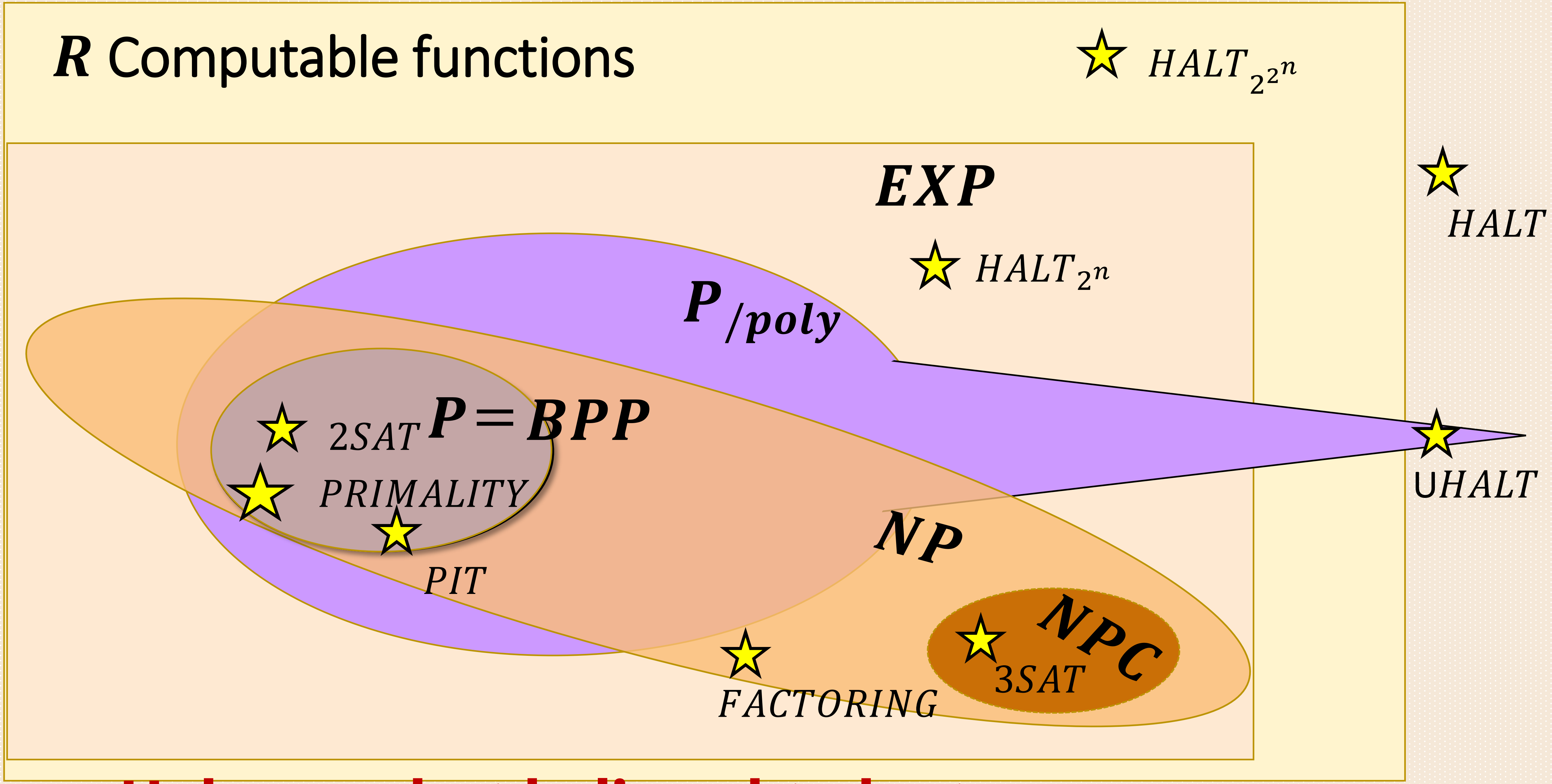
$$\Pr_{r\sim\{0,1\}^{q(n)}}[A(x;r) = F(x)] \geq \frac{2}{3}$$



All functions $F: \{0,1\}^* \to \{0,1\}$



All functions $F: \{0,1\}^* \to \{0,1\}$



Unknown but believed to be true

Next Lecture

- BPP vs EXP
- BPP vs P/poly
- BPP vs NP