CS 121 Section 10:
Poly-time Reductions, NP, and the Cook Levin Thm.

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Context for today

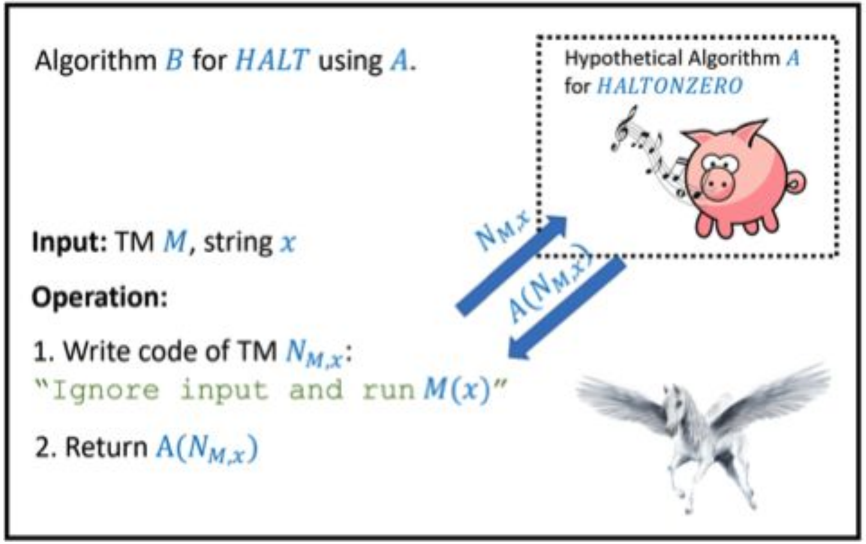
- Where did we come from?
  - P vs. EXP
  - Uncomputability: reductions and Rice’s Theorem
- Where are we now?
  - Revisiting reductions with the added notion of time complexity
    - Chapter 14: “Polynomial time reductions” (link to text)
  - NP, NP completeness, and Cook Levin Theorem
    - Chapter 15 (link to text)
- Where are we going?
  - Explore more classes of Functions and algorithms
    - Randomized algorithms

Polynomial-time Reductions

Past: Reductions to prove Uncomputability

In order to prove that *HALTONZERO* is uncomputable, we show *HALT* ≤ *HALTONZERO* :

1. Compute *HALT* using *HALTONZERO*
   a. *HALT*(M,x) = *HALTONZERO*(M)
   b. *HALT(M,x) = HALTONZERO(G(M,x))* where *G* is a reduction function to transform the inputs of *HALTONZERO* to *HALT*
2. Show correctness
   a. Soundness and completeness



Ex: Reducing *HALT* to *HALTONZERO*.

Polynomial-time Reductions

In order to show $A \leq B$:

1. Compute *A* using *B*
   a. *A*(M,x) = *B*(M)
   b. *A(M,x) = B(G(M,x))* where *G* is a reduction function to transform the inputs of *B* to *A*
2. Show correctness
   a. Soundness and completeness

In order to show $A \leq_p B$:

1. Compute *A* using *B*
   a. *A*(M,x) = *B*(M)
   b. *A(M,x) = B(R(M,x))* where *R* is a polynomial-time computable reduction function that transforms the inputs of *B* to *A*
2. Show correctness
   a. Soundness and completeness
   b. Show *R* can be computed in polynomial-time

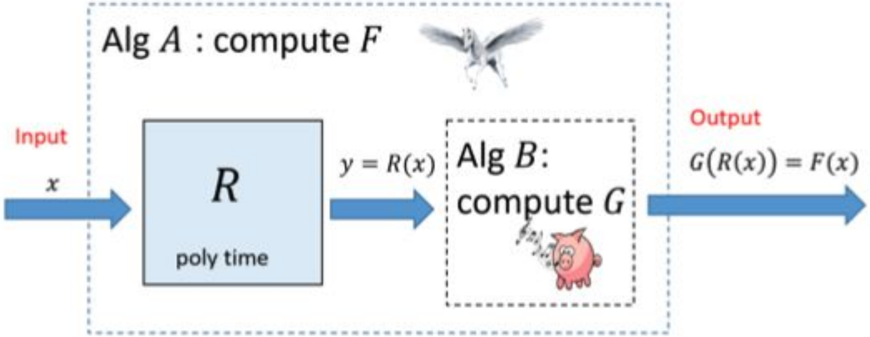
Formal Definition

Definition 14.1 (Polynomial-time reductions)

Let $F, G : \{0,1\}^* \rightarrow \{0,1\}$. We say that F *reduces to* G, denoted by $F \leq_p G$ if there is a polynomial-time computable $R : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every $x \in \{0,1\}^*$,

$$F(x) = G(R(x)) \ . \quad (14.2)$$

We say that F and G have *equivalent complexity* if $F \leq_p G$ and $G \leq_p F$.



Why do we care?

Proving $A \leq_p B$ leads to this stipulation:

If B can be computed with a poly-time algorithm,
Then A is also computable in poly-time.

(Concept check: why can't we say this with the regular reduction $A \leq B$?)

NP

Class of NP Functions

Definition 15.1 (NP)

We say that $F : \{0,1\}^* \to \{0,1\}$ is in $\mathbf{NP}$ if there exists some integer $a > 0$ and $V_F : \{0,1\}^* \to \{0,1\}$ such that $V_F \in \mathbf{P}$ and for every $x \in \{0,1\}^n$,

$$F(x) = 1 \Leftrightarrow \exists_{w \in \{0,1\}^{n^a}} \text{ s.t. } V_F(xw) = 1 \,. \quad (15.1)$$

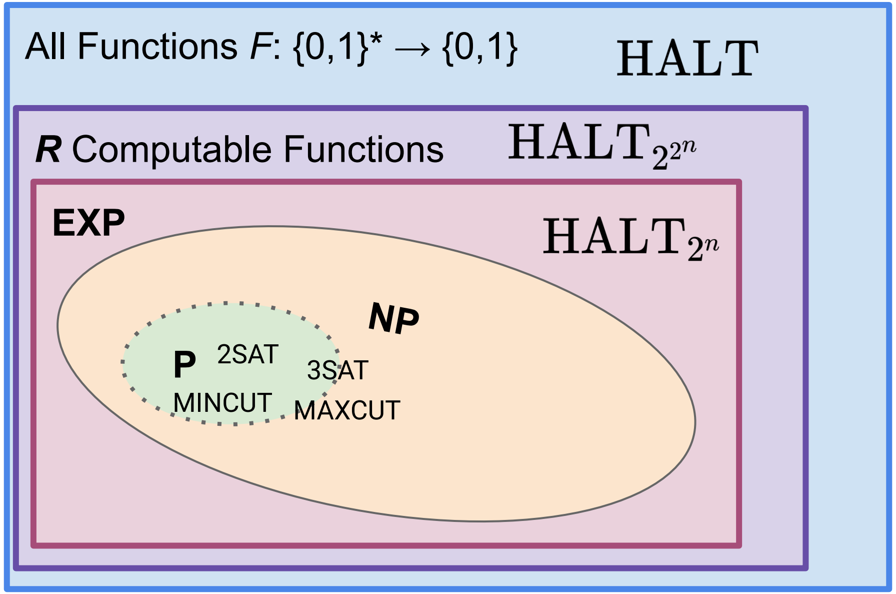
Non-mathematical explanation:

- **NP** is the class of functions that are efficiently verifiable.

NP := Non-deterministic Polynomial-time

NP ≠ Not Polynomial-time

Concept Check: Prove NP $\subseteq$ EXP



NP Hardness and Completeness

NP-hard

- A function *F* is NP-hard if all functions in NP can be reduced to it
  - *F* is "at least as hard" as all functions in NP; $\forall G \in \mathrm{NP},\ G \leq_p F$

NP-complete

- A function *F* is NP-complete if
  1. *F* is in NP
  2. *F* is NP-hard

Cook–Levin Theorem

Cook–Levin Theorem

Once we prove $A \leq_p B$, we know that:

- If *B* is in NP then *A* is also in NP.
- If *A* is NP-hard then *B* is also NP-hard.

Theorem 15.6 (Cook-Levin Theorem)

For every $F \in \mathbf{NP}$, $F \leq_p 3SAT$.

=> *3SAT* is NP-complete

Key Concept: Given *F* and *x*, we use V_F and *x* to get a small circuit *C* s.t. $C(w) = 1$ iff $V_F(x,w) = 1$.

Example Reductions

Zero-One Linear Equations Problem

Problem: show that there is no efficient polynomial-time algorithm to compute the *Zero-One Linear Equations Problem*.

The *Zero-One Linear Equations Problem* corresponds to the function $01EQ : \{0,1\}^* \rightarrow \{0,1\}$ where the input is a collection $= E$ of linear equations in variables $x_0, ...x_{n-1}$, and the output is 1 iff $\exists$ assignment $x \in \{0,1\}^n$ of 0/1 values to the variables that satisfies all the equations. So if E encodes the equations $x_0 + x_1 + x_2 = 2, x_0 + x_2 = 1, x_1 + x_2 = 2$ then $01EQ = 1$ because there exists an assignment to satisfy this ($x = 011$).

Ideas?

Zero-One Linear Equations Problem

Problem: show that there is no efficient polynomial-time algorithm to compute the *Zero-One Linear Equations Problem*.

To prove this, we can reduce a known NP-Hard problem (i.e. 3SAT) to Zero-One Linear Equations to show 3SAT $\leq_p$ 01EQ.

Steps for a Poly-time Reduction Proof

1. Describe a reduction function R to transform the inputs
2. Show R can be computed in polynomial time
3. Show Correctness
   a. Completeness
   b. Soundness

Step 1: Describe a reduction function R

We want to convert the inputs of 3SAT to the inputs of 01EQ

$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ -> system of 01 linear equations

We need to constrain each variable.

Hint: make each clause an equation. Any ideas on approaching this?

Step 1: Describe a reduction algorithm R

We want to convert the inputs of 3SAT to the inputs of 01EQ

$(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ -> system of 01 linear equations

Proof idea: A constraint $x_2 \vee \bar{x_5} \vee x_7$ can be rewritten as $x_2 + (1 - x_5) + x_7 \geq 1$.
Making it an equation: Since the sum of the left hand side is an inequality but can be at most 3, we add *auxiliary variables* to make it an equality.
Dealing with negated variables: We also add another variable x'_i to correspond to the negation of x_i and include the equation $x_i + x'_i = 1$.
Thus we have transformed

$$x_2 \vee \bar{x_5} \vee x_7 \implies x_2 + x'_5 + x_7 + y + z = 3$$

More generically, we transform:

$$x_1 \vee x_2 \vee x_3 \implies x_1 + x_2 + x_3 + u + v = 3$$

Step 2: Show R runs in polynomial time

- Initial loop of n steps to set up constraint for each variable
  - Each iteration takes constant time
- Another loop of m steps
  - Each iteration also taking constant time to convert a clause into an equation

Step 3a: Show Completeness

Step 3a: Show completeness: If $3SAT(\varphi) = 1$ **then** $01EQ(R(\varphi) = 1$

This is the first part of our proof of correctness. Suppose that $3SAT(\varphi) = 1$, then there is an assignment w to satisfy φ. Every clause in φ has form $w_1 \vee w_2 \vee w_3$ so because $w_1 + w_2 + w_3 \geq 1$ we can represent it as $w_1 + w_2 + w_3 + y + z = 3$ where y and z are between 0 and 1. If we let $x'_i = 1 - x_i$ for each variable i, the assignment $x_0 \cdots x_{n-1}, x'_0 \cdots x'_{n-1}, y_0 \cdots y_{m-1}, z_0 \cdots z_{m-1}$ satisfies $E = R(\varphi)$ so $01EQ(R(\varphi)) = 1$.

Step 3b: Show soundness

Step 3b: Show soundness: If $01EQ(R(\varphi) = 1$ then $3SAT(\varphi) = 1$

This is the second part of our proof of correctness. Suppose that $01EQ(R(\varphi) = 1$. Then there must be some assignment $x_0 \cdots x_{n-1}, x'_0 \cdots x'_{n-1}, y_0 \cdots y_{m-1}, z_0 \cdots z_{m-1}$.

Based on the way we did our transformation R, we know that x'_i is the negation of x_i for all $i \in [n]$. Because we defined $y_j, z_j \in [0, 1]$, $y_j + z_j \leq 2$ for all j in $[m]$. Thus, for every clause C_j in φ of the form $w_1 \vee w_2 \vee w_3$, we have $w_1 + w_2 + w_3 \geq 1$. This means the assignment $x_0 \cdots x_{n-1}$ satisfies φ and thus $3SAT(\varphi) = 1$.

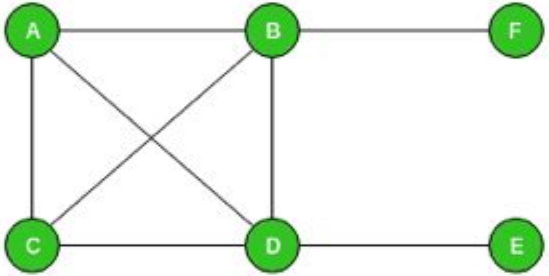
Another Example

Clique

Definition: Given an undirected graph $G = (V, E)$, a clique is a subset $V' \subseteq V$ *s.t.* $(v_1, v_2) \in E$ for all $v_1, v_2 \in V'$.

Consider the function *CLIQUE(G, k)* = 1 iff G has a clique of size k, and 0 otherwise.

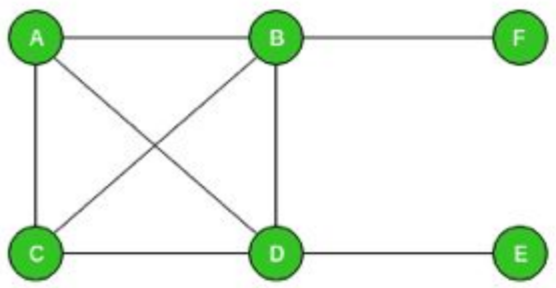
Is *CLIQUE* **NP**-complete? Prove it or show why not.



Clique

Reminder: To prove NP-completeness, we need to:

1. Show *CLIQUE* $\in$ NP



2. Show *CLIQUE* is NP-hard by reducing another NP-hard problem to *CLIQUE* (e.g. 3SAT $\leq_p$ *CLIQUE*).

Clique

Reminder: To prove NP-completeness, we need to:

1. Show *CLIQUE* $\in$ NP.
2. Show *CLIQUE* is NP-hard by reducing another NP-hard problem to *CLIQUE* (e.g. 3SAT $\leq_p$ *CLIQUE*).

Some NP-hard problems:

1. 3SAT
2. Min. k-cut
3. ISET
4. HALT

Steps for a Poly-time Reduction Proof

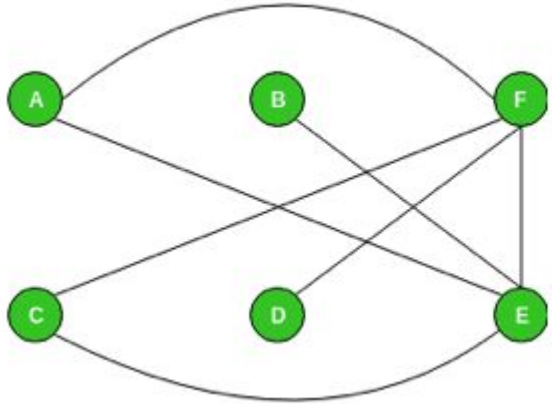
1. Describe a reduction function R to transform the inputs
2. Show R can be computed in polynomial time
3. Show Correctness
   a. Completeness
   b. Soundness

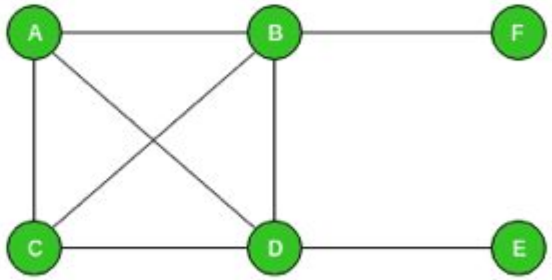
Step 1: Describe a reduction function R

We want to convert the inputs of *ISET* to the inputs of *CLIQUE*

ISET(G(V,E), k)=1 iff G contains an indep. set of size ≥ k.

CLIQUE(G’(V’,E’), k)=1 iff G’ contains a clique of size ≥ k.



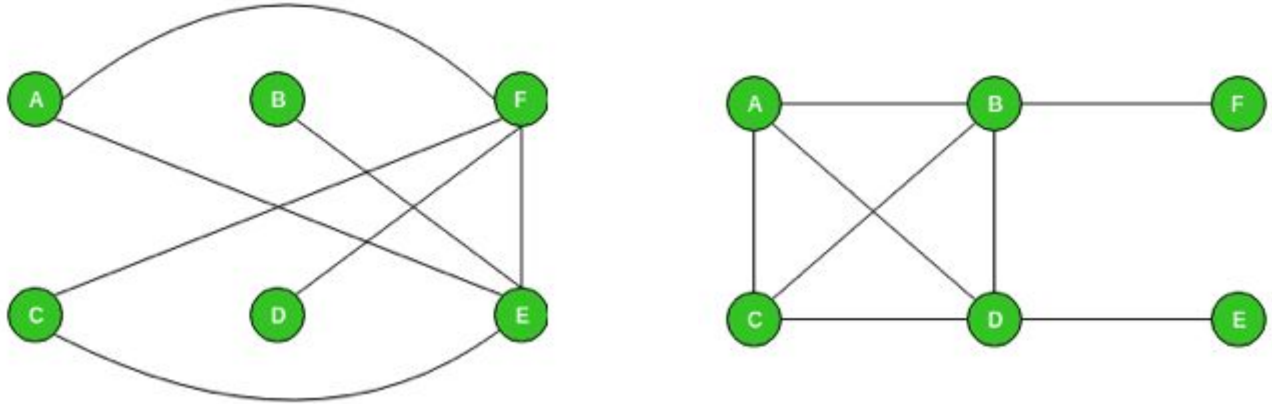


Step 1: Describe a reduction function R

We want to convert the inputs of *ISET* to the inputs of *CLIQUE*

ISET(G(V,E), k)= CLIQUE(R(G,k))

where R(G,k) returns the complement of G and keeps k the same

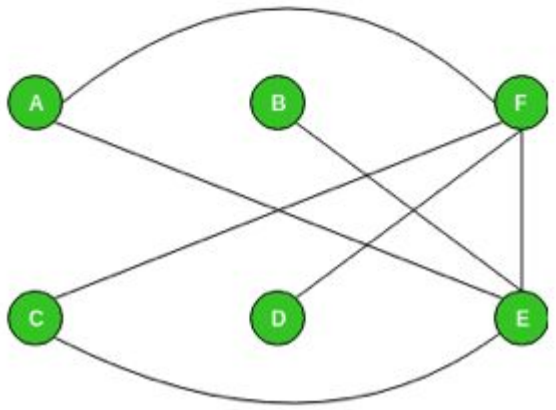


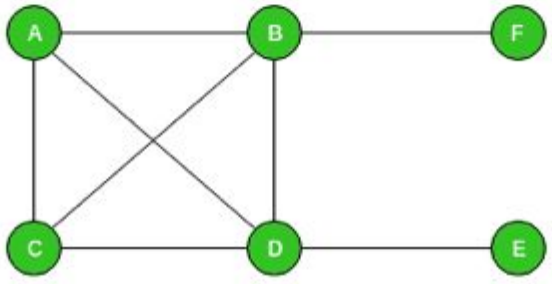
Step 2: Show R runs in polynomial time

- Converting G to its complement G’ takes O(|V| + |E|) time.

Step 3a: Show Completeness

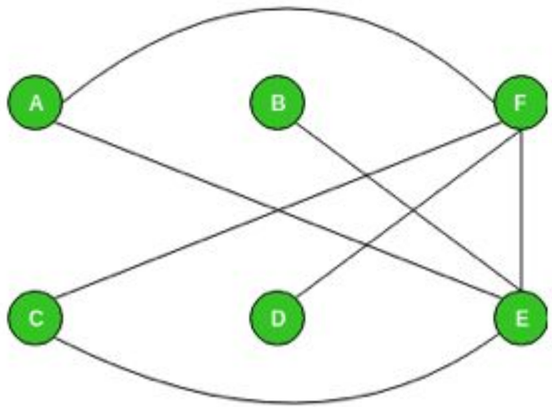
Show: if *ISET*(G, k) = 1 then *CLIQUE(G'*, k) = 1.

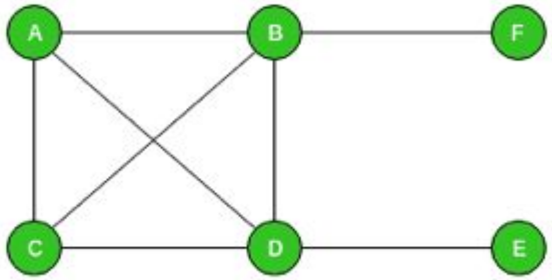




Step 3b: Show soundness

Show: if *CLIQUE*(G', k) = 1 then *ISET*(G, k) = 1.

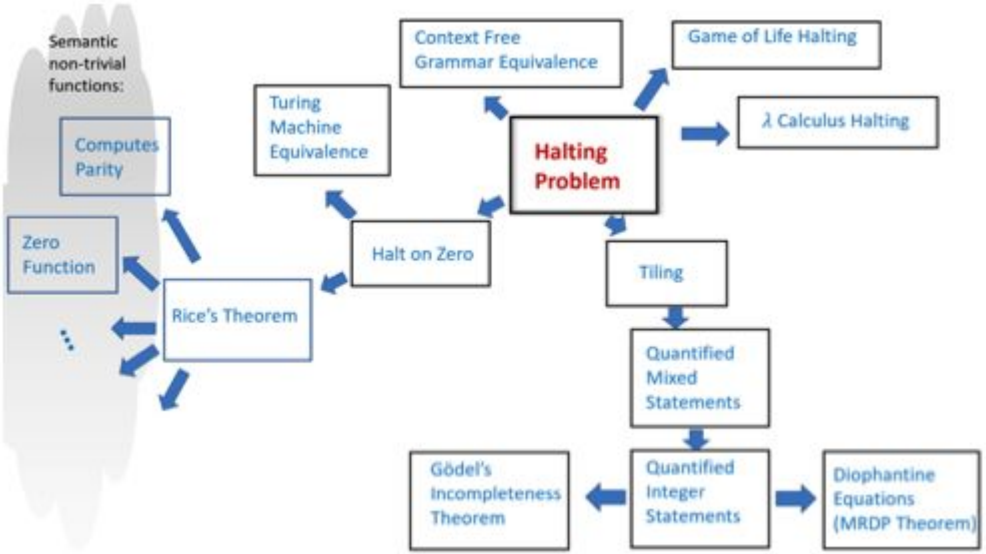


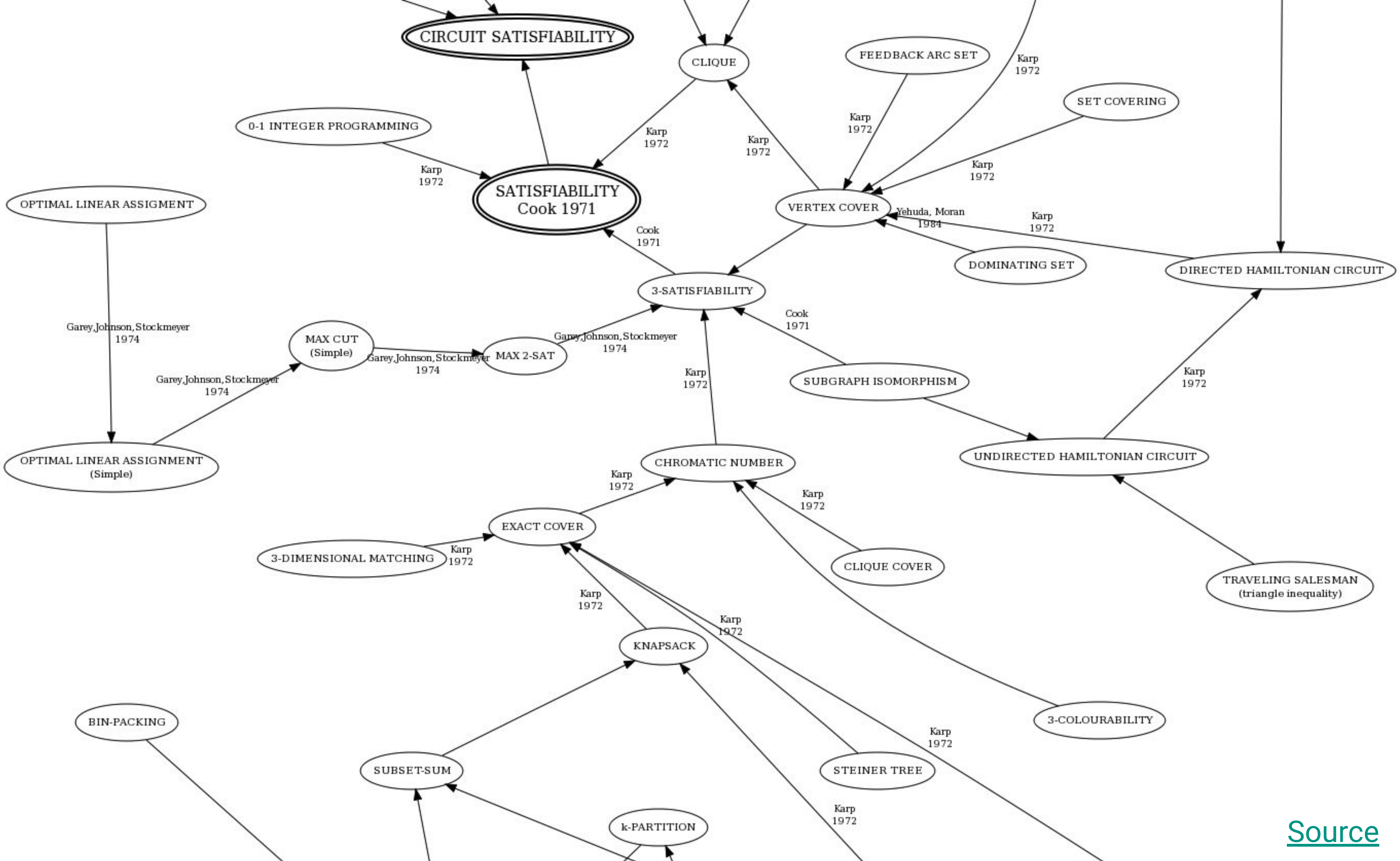


Implications of All of This

Reminder: Uncomputability

Once proven $A \leq B$: If A is uncomputable then B is also uncomputable.




CIRCUIT SATISFIABILITY
CLIQUE
FEEDBACK ARC SET
Karp
1972
SET COVERING
0-1 INTEGER PROGRAMMING
Karp
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Karp
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Karp
1972
Karp
1972
Karp
1972
SATISFIABILITY
Cook 1971
OPTIMAL LINEAR ASSIGMENT
VERTEX COVER
Yehuda, Moran
1984
Karp
1972
Cook
1971
DOMINATING SET
DIRECTED HAMILTONIAN CIRCUIT
3-SATISFIABILITY
Cook
1971
Garey,Johnson,Stockmeyer
1974
MAX CUT
(Simple)
Garey,Johnson,Stockmeyer
1974
MAX 2-SAT
Garey,Johnson,Stockmeyer
1974
Karp
1972
SUBGRAPH ISOMORPHISM
Karp
1972
Garey,Johnson,Stockmeyer
1974
OPTIMAL LINEAR ASSIGNMENT
(Simple)
CHROMATIC NUMBER
UNDIRECTED HAMILTONIAN CIRCUIT
Karp
1972
Karp
1972
EXACT COVER
3-DIMENSIONAL MATCHING
Karp
1972
CLIQUE COVER
TRAVELING SALESMAN
(triangle inequality)
Karp
1972
Karp
1972
KNAPSACK
BIN-PACKING
3-COLOURABILITY
Karp
1972
SUBSET-SUM
STEINER TREE
k-PARTITION
Karp
1972

What if P = NP?

- Difference between proving P = NP and actually discovering / constructing a “fast” algorithm
- Public-key cryptography
  - Passwords are verifiable in poly-time
- Automating discovery of mathematical proofs
- Videos to check out
  - Richard Karp on difficulty of proving P=NP
  - Donald Knuth’s intuition on P=NP
  - Scott Aaronson on what happens if P=NP

More Problems

1. Transitivity of Poly-time Reductions

Show that for every $F, G, H : \{0,1\}^* \to \{0,1\}$, if $F \leq_p G$ and $G \leq_p H$, then $F \leq_p H$.

2. Vertex–Cover

Given an undirected graph $G = (V, E)$, a vertex-cover is a subset $V' \subseteq V$ *s.t.* for all $(v_1, v_2) \in E$, either $v_1 \in V'$ or $v_2 \in V'$. Consider the function *VERTEX-COVER(G, k)* = 1 iff G has a vertex cover of size k, and 0 otherwise. Is *VERTEX-COVER* **NP**-complete? Prove it or show why not.

3. Set-Cover NP-Completeness

Given n sets $S_1, S_2, \ldots, S_n$ such that

$$\bigcup_{i=1}^{n} S_i = A$$

the set cover of size k over these sets is a collection C of k of these sets such that

$$\bigcup_{i \in C} S_i = A$$

Given a collection of sets and an integer k, SET-COVER returns if there exists a valid set cover of a most size k over the given collection of sets. Prove that SET-COVER is NP-complete.

4. Say if P, NP, or Uncomputable:

(a) Given an integer x, determine if x has a prime factor that is at most k.

(b) Given an undirected graph graph, determine whether it is possible to partition its vertices into two sets, with at least k edges crossing between sets.

(c) Given a program Q, an input x, and a string 1^t, determine whether Q halts on x within t steps.

Thank You!

Appendix

coNP

Show that coNP = NP

Define $F \in \mathsf{coNP}$ iff $\overline{F} \in \mathsf{NP}$, where $\overline{F}$ denotes the negation of the output of F (for example, if $F(00) = 1$, then $\overline{F}(00) = 0$). Prove that if $\mathsf{P} = \mathsf{NP}$, then $\mathsf{coNP} = \mathsf{NP}$.