Problem 1. For each of the following sets of numbers, add the letters to the tree that is the most efficient representation:

\{E, H, G, K, Z\}, \{O, I, N, D, C\}, \{T, L, U, M, P\}.

Solution 1:
**Problem 2.** Show that we can transform any representation to a prefix-free one by a modification that takes a \( k \) bit string into a string of length at most \( k + \log k + O(\log \log k) \).

**Solution 2:**

*Proof.* Assume the veracity of the practice problem (which was originally part (a) of this problem): that we can transform any representation to a prefix-free one that takes a \( k \) bit string of length at most \( k + O(\log k) \).

Then, we can the same technique proven effective before, but instead of encoding the length using the standard C-style transformation, we use the transformation we assume: one that takes a \( k \) bit string of length at most \( k + O(\log k) \).

Thus, our new transformation uses \( (\log k) + O(\log(\log k)) \) bits to represent the length, and \( k \) bits to replicate the string: in total, \( k + \log k + O(\log \log k) \) bits, as desired.

**Problem 3.**

Let \( NtS \) be the function \( NtS: \mathbb{N} \to \{0,1\}^* \) that sends a number to its binary representation; more explicitly:

\[
NtS = \begin{cases} 
0 & n = 0 \\
1 & n = 1 \\
NtS(\lfloor n/2 \rfloor) \text{parity}(n) & n > 1,
\end{cases}
\]

where \( \text{parity} : \mathbb{N} \to \{0,1\}^* \) is the function defined as \( \text{parity}(n) = 0 \) if \( n \) is even and \( \text{parity}(n) = 1 \) if \( n \) is odd, and as usual, for strings \( x, y \in \{0,1\}^* \), \( xy \) denotes the concatenation of \( x \) and \( y \).

(a) Prove that \( NtS \) satisfies that for every \( n \in \mathbb{N} \), if \( x = NtS(n) \) then \( |x| = 1 + \max(0, \lfloor \log_2 n \rfloor) \).

(b) Prove that \( NtS \) is in fact a representation, by showing that it is one-to-one: that is, find a function \( StN: \{0,1\}^* \to \mathbb{N} \) such that \( StN(NtS(n)) = n \).

**Solution 3:**

*Proof.*

(a) We will prove by (strong) induction that \( NtS \) satisfies that for every \( n \in \mathbb{N} \), if \( x = NtS(n) \) then \( |x| = 1 + \max(0, \lfloor \log_2 n \rfloor) \).

It’s clear immediately from the definition that the base cases \( x = 0 \) and \( x = 1 \) satisfy \( |x| = 1 + \max(0, \lfloor \log_2 n \rfloor) \).

For \( n \geq 1 \) we have that \( \lfloor \log_2 n \rfloor \geq 0 \), so we must prove that \( |x| = \lfloor \log_2 n \rfloor \) exactly.

Suppose that we have some \( n \in \mathbb{N} \) and \( x = NtS(n) \). We have that \( NtS(n) = NtS(\lfloor n/2 \rfloor) \text{parity}(n) \).

We assume as our inductive hypothesis that the length of \( NtS(m) \) for all \( m < n \) (with \( m \neq 0,1 \)) is \( \lfloor \log_2 m \rfloor \).

Thus we have that \( NtS(n) = NtS(\lfloor n/2 \rfloor) \text{parity}(n) \) has length

\[
\left\lfloor \log_2 \left( \lfloor n/2 \rfloor \right) \right\rfloor + 1 = \left\lfloor \log_2 n \right\rfloor - 1 + 1 = \left\lfloor \log_2 n \right\rfloor,
\]

as desired.
(b) We will prove that $NtS$ is in fact a representation, by showing that it is one-to-one: that is, we will find a function $StN : \{0, 1\}^* \to \mathbb{N}$ such that $StN(NtS(n)) = n$.

Let $StN : \{0, 1\}^* \to \mathbb{N}$ be the function that maps some string $x$ to

$$\sum_{i \in [||x||]} x[i]2^{||x||-1-i}.$$ 

We will prove by induction that $StN(NtS(n)) = n$.

We will first observe that

$$StN(NtS(0)) = StN(0) = 0,$$

and that

$$StN(NtS(1)) = StN(1) = 1 \cdot 2^0 = 1,$$

as desired.

Then assume for strong induction that $StN(NtS(m)) = m$ for $m < n$. We will prove that $StN(NtS(n)) = n$, which will conclude our proof.

Let $NtS(n) = x$. Note that $|NtS([n/2])| = |x| - 1$.

We have

$$StN(NtS(n)) = StN(NtS([n/2]) \text{parity}(n))$$

$$= \sum_{i \in [||x||-1]} x[i]2^{||x||-1-i} + \text{parity}(n)2^{||x||-1-|x|+1}$$

$$= 2 \cdot StN(NtS([n/2])) + \text{parity}(n)$$

$$= 2 \cdot \lfloor n/2 \rfloor + \text{parity}(n)$$

$$= n,$$

as desired.