Section 1 Solutions

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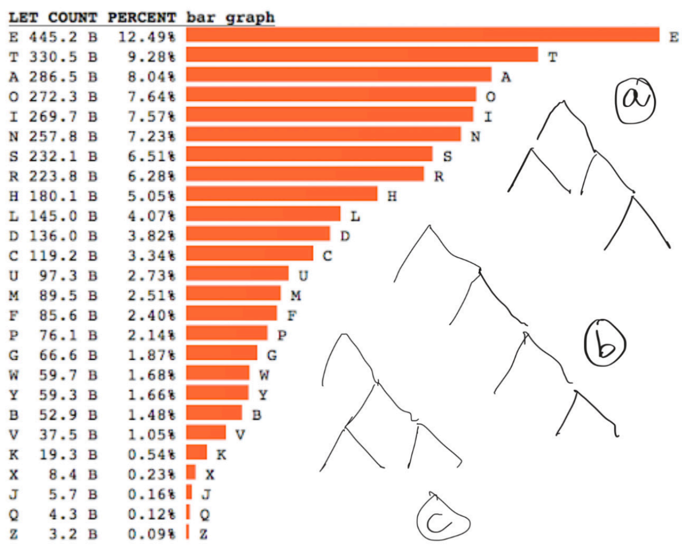
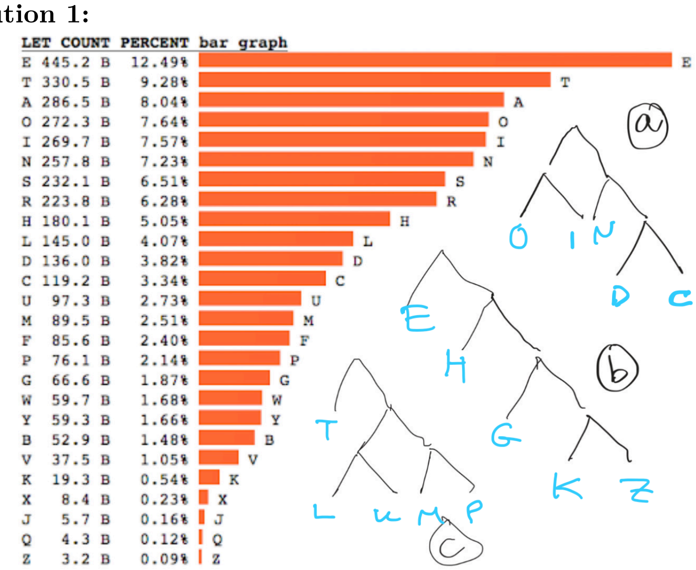
Problem 1. For each of the following sets of numbers, add the letters to the tree that is the most efficient representation:

{E, H, G, K, Z}, {O, I, N, D, C}, {T, L, U, M, P}.



Solution 1:



Problem 2. Show that we can transform any representation to a prefix-free one by a modification that takes a k bit string into a string of length at most $k + \log k + O(\log \log k)$.

Solution 2:

Proof. Assume the veracity of the practice problem (which was originally part (a) of this problem): that we can transform any representation to a prefix-free one that takes a k bit string of length at most $k + O(\log k)$.

Then, we can the same technique proven effective before, but instead of encoding the length using the standard C-style transformation, we use the transformation we assume: one that takes a k bit string of length at most $k + O(\log k)$.

Thus, our new transformation uses $(\log k) + O(\log(\log k))$ bits to represent the length, and k bits to replicate the string: in total, $k + \log k + O(\log \log k)$ bits, as desired.

□

Problem 3.

Let NtS be the function NtS: $\mathbb{N} \to \{0,1\}^*$ that sends a number to its binary representation; more explicitly:

$$NtS = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ NtS(\lfloor n/2 \rfloor)parity(n) & n > 1, \end{cases}$$

where $parity : \mathbb{N} \to \{0,1\}^*$ is the function defined as $parity(n) = 0$ if n is even and $parity(n) = 1$ if n is odd, and as usual, for strings $x, y \in \{0,1\}^*$, xy denotes the concatenation of x and y.

(a) Prove that NtS satisfies that for every $n \in \mathbb{N}$, if $x = NtS(n)$ then $|x| = 1 + \max(0, \lfloor \log_2 n \rfloor)$.

(b) Prove that NtS is in fact a representation, by showing that it is one-to-one: that is, find a function $StN : \{0,1\}^* \to \mathbb{N}$ such that $StN(NtS(n)) = n$.

Solution 3:

Proof.

(a) We will prove by (strong) induction that NtS satisfies that for every $n \in \mathbb{N}$, if $x = NtS(n)$ then $|x| = 1 + \max(0, \lfloor \log_2 n \rfloor)$.

It's clear immediately from the definition that the base cases $x = 0$ and $x = 1$ satisfy $|x| = 1 + \max(0, \lfloor \log_2 n \rfloor)$.

For $n \geq 1$ we have that $\lfloor \log_2 n \rfloor \geq 0$, so we must prove that $|x| = \lfloor \log_2 n \rfloor$ exactly.

Suppose that we have some $n \in \mathbb{N}$ and $x = NtS(n)$. We have that $NtS(n) = NtS(\lfloor n/2 \rfloor)parity(n)$. We assume as our inductive hypothesis that the length of $NtS(m)$ for all $m < n$ (with $m \neq 0, 1$) is $\lfloor \log_2 m \rfloor$.

Thus we have that $NtS(n) = NtS(\lfloor n/2 \rfloor)parity(n)$ has length

$$\begin{aligned} \lfloor \log_2(\lfloor n/2 \rfloor) \rfloor + 1 &= \lfloor \log_2 n \rfloor - 1 + 1 \\ &= \lfloor \log_2 n \rfloor, \end{aligned}$$

as desired.

(b) We will prove that NtS is in fact a representation, by showing that it is one-to-one: that is, we will find a function $StN : \{0,1\}^* \to \mathbb{N}$ such that $StN(NtS(n)) = n$.

Let $StN : \{0,1\}^* \to \mathbb{N}$ be the function that maps some string x to

$$\sum_{i \in [|x|]} x[i] 2^{|x|-1-i}.$$

We will prove by induction that $StN(NtS(n)) = n$.

We will first observe that

$$\begin{aligned} StN(NtS(0)) &= StN(0) \\ &= 0, \end{aligned}$$

and that

$$\begin{aligned} StN(NtS(1)) &= StN(1) \\ &= 1 \cdot 2^0 \\ &= 1, \end{aligned}$$

as desired.

Then assume for strong induction that $StN(NtS(m)) = m$ for $m < n$. We will prove that $StN(NtS(n)) = n$, which will conclude our proof.

Let $NtS(n) = x$. Note that $|NtS(\lfloor n/2 \rfloor)| = |x| - 1$.

We have

$$\begin{aligned} StN(NtS(n)) &= StN(NtS(\lfloor n/2 \rfloor) parity(n)) \\ &= \sum_{i \in [|x|-1]} x[i] 2^{|x|-1-i} + parity(n) 2^{|x|-1-|x|+1} \\ &= 2 \cdot StN(NtS(\lfloor n/2 \rfloor)) + parity(n) \\ &= 2 \cdot \lfloor n/2 \rfloor + parity(n) \\ &= n, \end{aligned}$$

as desired.

□