CS 121 Section 7:
Universal TMs, Uncomputability

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Today's Topics

1. Universality
2. Existence of an Uncomputable Function
3. Uncomputability of HALT
4. Reductions
5. Example Uncomputability Proof

Universality (Circuits)

1. Applied to Boolean circuits and straight-line programs
2. Circuit to evaluate all other circuits with a caveat*
3. *Needed a universal circuit of >s gates to evaluate a circuit of s gates

Universality (TMs)

1. Applies to TMs and their equivalents (e.g. NAND-TM)
2. Turing machine U to evaluate all other Turing machines
3. Can evaluate Turing Machines that are more complex than U using U

Formal Definition of Universal TM

Theorem 9.1: Universal Turing Machine

There exists a Turing machine U such that on every string M which represents a Turing machine and $x \in \{0,1\}^*, U(M,x) = M(x)$

Alternatively, U outputs what TM M would on input x, given a first argument M and second argument x. If the M on input *x* does not output anything (does not halt), neither will *U(M, x)*.

Proved by construction.

Existence of Uncomputable Functions

Uncomputable: **function** that cannot be computed by any **Turing machine.**

- Extra important now to differentiate functions vs. programs/Turing machines

Theorem 9.5: Uncomputable Functions
There exists a function $F^* : \{0,1\}^* \to \{0,1\}$ that is not computable by any Turing machine.

Proof Idea: Cantor's proof that the reals are uncountable.

Prove that a function F^* is uncomputable

Define $G(x)$

- If x represents a valid TM M and $M(x)$ halts, $G(x)$ = first bit of $M(x)$
- In any other case of x, $G(x) = 1$

$F^* = 1 - G(x)$

Proof Idea: contradiction. If F* were computable, there is a M that computes it and halts on all inputs. Let x be the string representation of M. $G(x)$ = the first bit of $M(x)$ by construction. So, $F^*(x) = 1$ - first bit of $M(x)$, again by construction. But we just said M computes F^*. How can their outputs differ in the first bit on input x? Contradiction.

What is HALT?

***HALT(M, x)* = 1** if Turing machine *M* halts on the input *x*

***HALT(M, x)* = 0** otherwise.

(You might object that you can compute HALT(*M*, *x*) by simulating *M*. But, if you tried to write such a program P, after what point could P definitively say *M* will never finish? (Never.))

Uncomputability of *HALT*

Proof Idea: Use the fact that F^* is uncomputable.

1. We know F^* is uncomputable.
2. Assume, toward a contradiction, that *HALT* is computable.
3. **Show that the program that computes *HALT*, which exists since we assumed *HALT* is computable, enables us to compute F^*.**
4. But #1 says F^* is uncomputable. So we have a contradiction :(
5. Our assumption in #2 therefore must be wrong. *HALT* is uncomputable.

Step 3 in Detail

Key Idea: If program P that computed *HALT* existed, it would help us build a program P* that computes *F**.

$F^* : \{0,1\}^* \rightarrow \{0,1\}$

- If x represents a valid TM *M* and *M(x)* halts, *F*(x)* = 1- first bit of *M(x)*
- In any other case of x, *F*(x) = 0*

$HALT : \{0,1\}^* \rightarrow \{0,1\}$

- *1 iff M* halts on input *x*

Input: $x \in \{0,1\}^*$
Output: $F^*(x)$
$PHALT$ is the program that computes $HALT$
U is the universal TM

```
def PFSTAR(x):
    if (PHALT(x, x) = 0):
        return 0
    if (U(x, x) = 0):
        return 1
    return 0
```

Proving Uncomputability via Reduction

Main Idea: Use the fact that function A is uncomputable to prove B is uncomputable.

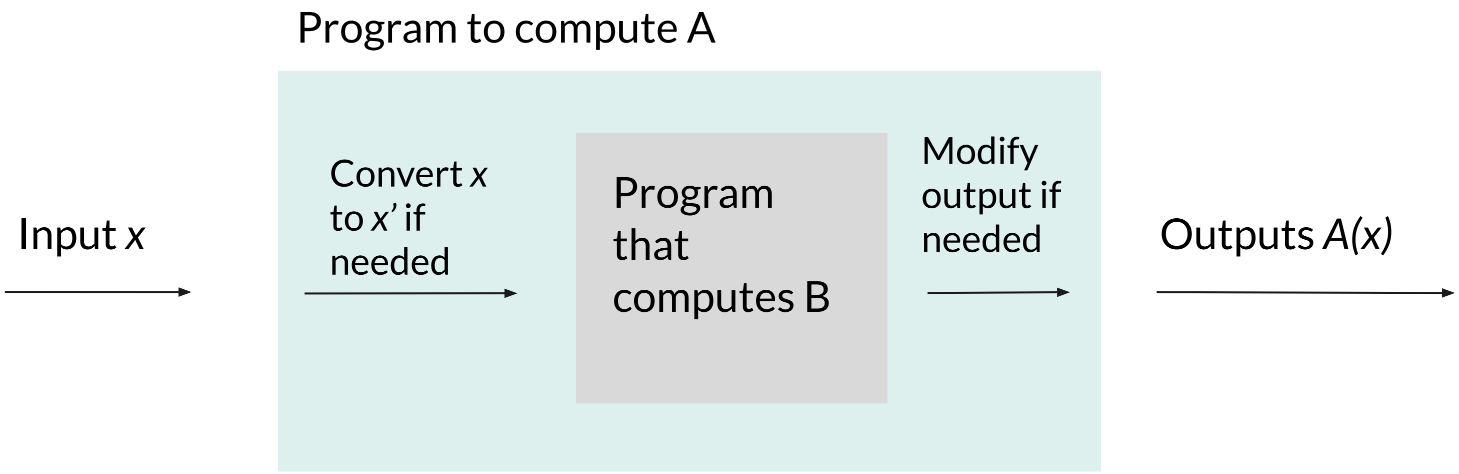
The "reduction": If we could compute B, we can compute A too, since the task of computing A *reduces* to just the task of computing B."

Reductions

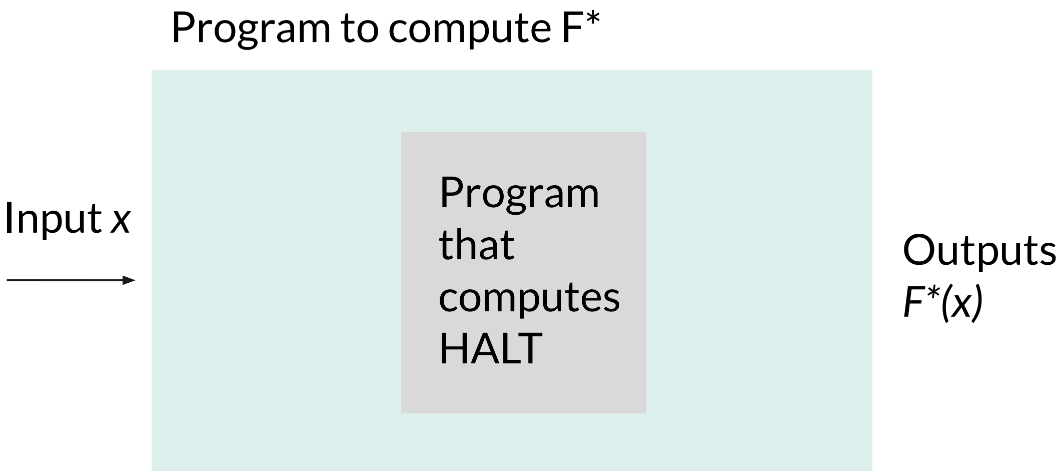
A = function we know is uncomputable. *B* = function we want to prove is uncomputable.

1. We know *A* is uncomputable.
2. Assume, toward a contradiction, that *B* is computable.
3. **Show that the program that computes *B*, which exists since we assumed *B* is computable, enables us to compute *A*. [REDUCTION STEP]**
4. But #1 says *A* is uncomputable. So we have a contradiction :(
5. Our assumption in #2 therefore must be wrong. *B* is uncomputable.

Reductions (Generally)



Reduction (HALT uncomputability)



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$PHALT$ is the program that computes $HALT$
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```
def PFSTAR(x):
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    if (U(x, x) = 0):
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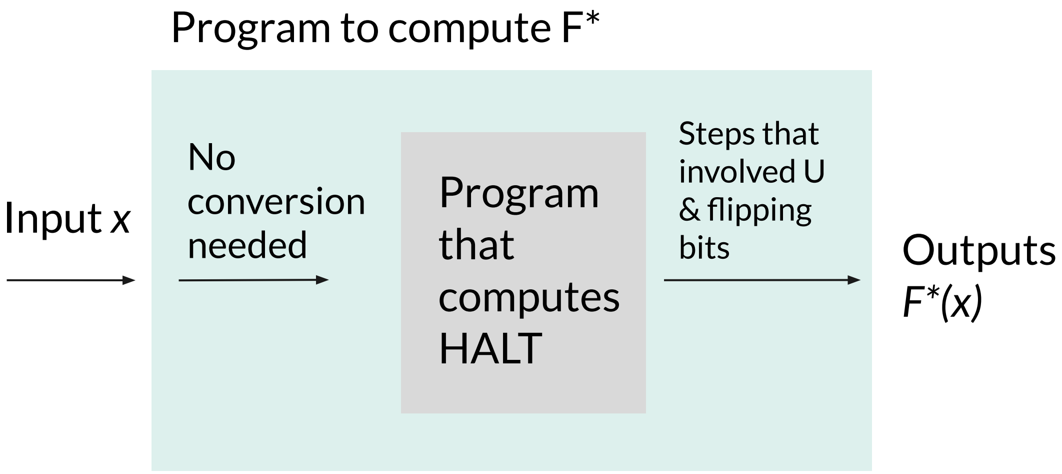
Reduction (HALT uncomputability)



Input: $x \in \{0,1\}^*$
Output: $F^*(x)$
$PHALT$ is the program that computes $HALT$
U is the universal TM

```
def PFSTAR(x):
    if (PHALT(x, x) = 0):
        return 0
    if (U(x, x) = 0):
        return 1
    return 0
```

Reduction from

A to B

We can write a program to compute this function if we can write a program to compute this function.

This is the function that we know is uncomputable, and this is the function we're proving is uncomputable.

We want to know how to compute/solve this, and we know how to compute/solve this.

We reduce the problem of computing this function to the problem of computing this function.

Example 1 F* HALT

Example 2 HALT HALTONZERO

Example: $HALTONZERO : \{0,1\}^* \to \{0,1\}$

- *HOZ(M) = 1* iff Turing machine represented by *M* halts on input 0

How can we prove uncomputability?

- Intuition: feels similar to *HALT*
- Use *HALT*'s uncomputability to prove *HOZ* is uncomputable

HALT

- Takes in *M* and *x*
- 1 iff M(x) halts

HOZ

- Takes in just *M*
- 1 iff M(0) halts

Example Uncomputability Proofs

Input: TM M, input x
Output: $HALT(M, x)$
$PHOZ$ is the program that computes HOZ

```
def PHALT(M, x):
    M' = description of the TM that takes an input, ignores it, and runs EVAL(M, x)

    return PHOZ(M')
```

Reduction (HOZ uncomputability)

Program
that
computes
HOZ
P(M')

Reduction (HOZ uncomputability)

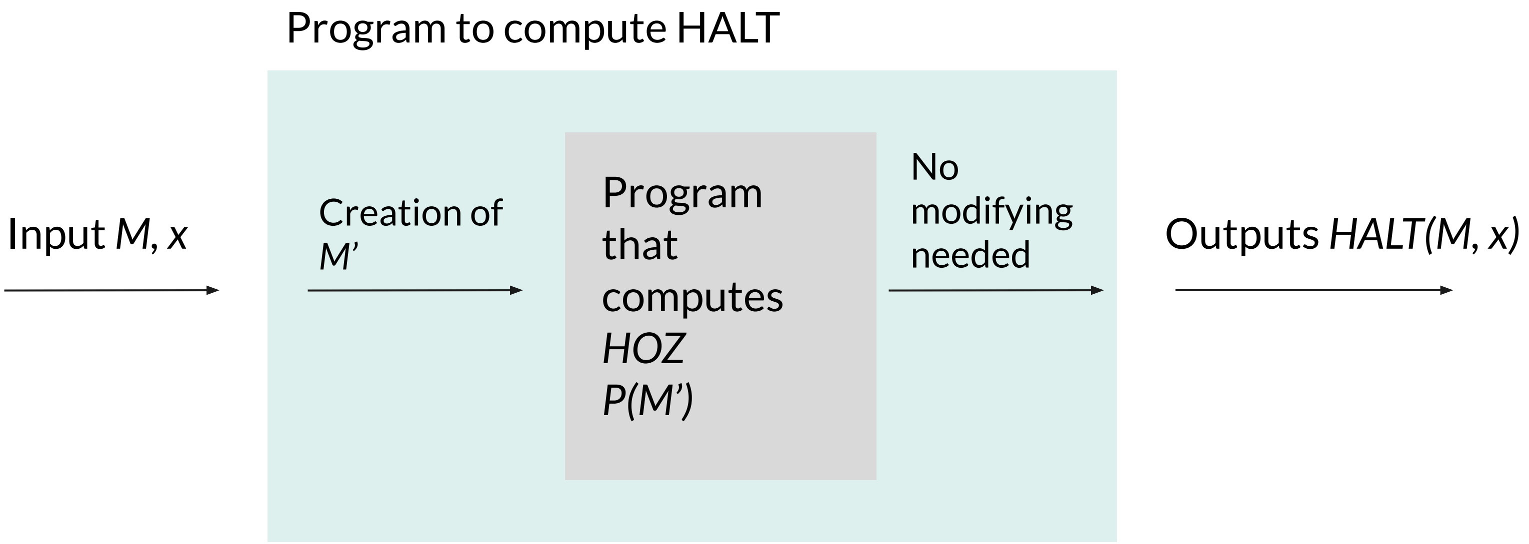
Program to compute HALT

Input *M*, *x*

Program that computes *HOZ P(M')*

Outputs *HALT(M, x)*

Reduction (HOZ uncomputability)



Uncomputability of *HOZ*

Proof Idea: Use the fact that *HALT* is uncomputable.

1. We know *HALT* is uncomputable.
2. Assume, toward a contradiction, that *HOZ* is computable.
3. **Show that the program that computes *HOZ*, which exists since we assumed *HOZ* is computable, enables us to compute *HALT*.**
4. But #1 says *HALT* is uncomputable. So we have a contradiction :(
5. Our assumption in #2 therefore must be wrong. *HOZ* is uncomputable.

Section Problems!

Let $TIMEDHALT : \{0,1\}^* \to \{0,1\}$ be the function that on input (a string representing) a triple (M, x, t), $TIMEDHALT(M, x, t) = 1$ iff the Turing machine M, on input x, halts within at most t steps (where a step is defined as one sequence of reading a symbol from the tape, updating the state, and writing a new symbol and (potentially) moving the head.)

Prove that $TIMEDHALT$ is computable.

Section Problems!

Let $\mathit{IS\text{-}TM\text{-}ONE}$: $\{0,1\}^* \to \{0,1\}$ be the function that takes as input a string representation of a Turing machine M and outputs 1 iff $M(x) = 1$ for every $x \in \{0,1\}^*$. Prove or disprove: $\mathit{IS\text{-}TM\text{-}ONE}$ is computable.

Section Problems!

Prove that the following function is uncomputable:

$$COMPUTES-PARITY(P) = \begin{cases} 1 & \text{P computes the parity function*} \\ 0 & \text{otherwise} \end{cases}$$

*Parity(x) = 1 iff x has an odd number of 1s.