Today
- Testing Tensor Product Codes
- Testing low degree Polynomials

Recall:
LDPC view of LTC:

- \( C_{\text{big}} = \{ (x_1, \ldots, x_n) \mid \forall j \in \mathbb{R} \ x|_{P(j)} \in C_{\text{Small}} \} \)

- Test: Pick random right vertex \( j \) and verify \( x|_{P(j)} \in C_{\text{Small}} \)

- \( \delta \): Soundness: \( \Pr[\text{reject} \ x] \geq \varepsilon \cdot S(x, C_{\text{big}}) \)

- \( \delta \)-Robustness: \( \frac{1}{\delta} \cdot S(x, C_{\text{big}}) \)

Test \( \delta \)-Robust \( \Rightarrow \) Test \( \delta \)-sound \( \Rightarrow \) Test \( \frac{\delta}{\delta} \)-robust.
\( C \times m \)

\( \begin{align*}
& x_1 \\
& \vdots \\
& \vdots \\
& x_N \\
N &= n^m
\end{align*} \)

Viderman Thm: \( \forall m \geq 3, \exists 0 < \alpha < 1 \) s.t. \( C \subset C^{m-1} \) test is for \( C^m \) is \( \alpha \)-robust.

Proof: will do \( m = 2 \) for simplicity. Idea works for \( m \geq 3 \).

Given: \( f: [n]^3 \rightarrow \mathbb{F}_2 \)

Let \( A, B, D: [n]^3 \rightarrow \mathbb{F}_2 \) be s.t.

1. \( \forall i \ A(i, \ldots, \cdot), \ B(\cdot, i, \cdot), \ D(\cdot, \cdot, i) \in C \times C \)

2. \( S(f, A) + S(f, B) + S(f, D) = \) local dist. of \( f \) from \( C \times C \times C \)

\[ \frac{3}{3} = \frac{1}{1} \]
Define \((i,j,k)\) to be \textit{Good} if
\[
A(i,j,k) = B(i,j,k) = D(i,j,k)
\]
Bad other wise.

Define \((i',j',k)\) etc. to be \textit{Bad} if
more than \(\frac{S^2 n^2}{2}\)
pairs points \((j,k)\) make \((i,j,k)\) Bad.

Claim: Bad-points live on Bad-Planes.

Proof: Suppose \(A(i,j,k) = B(i,j,k)\)
\[
\Rightarrow A(i,j,0) = B(i,j,0)
\Rightarrow S(A(i,j,0), B(i,j,0)) \geq S
\]
(since both are codewords of \(C\))

\[
\Rightarrow \text{for every } k' \text{ s.t.}
\]
\(A(i,j,k') = B(i,j,k')\) we have
Either \(A(i,j,k') = D(i,j,k')\) or
\(B(i,j,k') = D(i,j,k')\)

\[
\text{for } \geq \frac{SN}{2} \text{ k's.}
\]

Then \(A(i,0,k') = D(i,0,k')\) for \(\geq \frac{SN}{2} \text{ k's.}\)

\[
\Rightarrow \text{ for } \geq \frac{SN}{2} \text{ k's.}
\]

\[
\Rightarrow A \text{ is Bad.}
\]
Throw away i if \( A(i, \ldots) \) bad
\[ i \quad B(\ldots, j, \ldots) \text{ bad} \]
\[ j \quad D(\ldots, k) \text{ bad} \]

Remainings say \((i, j, k)\) excellent if \( A(i, \ldots) \) good
\[ B(\ldots, j, \ldots) \text{ good} \]
\[ D(\ldots, k) \text{ good} \]

Claim 1: # excellent points form \((1-\varepsilon)n \times (1-\varepsilon)n \times (1-\varepsilon)n\) cube.

Claim 2: if \((\delta)\) then \(A, B, D\) can be extended from excellent set \((\delta)\) to whole cube. \([n] \times [n] \times [n]\) together \(A, B, D\) C \& C \& C.

Claim 3: Extended function close to original function.

Proof of Claim 1: Reject rejects [bad point] \( \geq \frac{1}{3} \)
\[ \Rightarrow \text{ # bad points } \leq \]
\[ \Rightarrow \text{ Bad plane } A(i, \ldots) \text{ has } \frac{\varepsilon^2}{2} \text{ fraction bad points.} \]
\[ \Rightarrow \text{ fraction bad i planes } \leq \frac{3\varepsilon^2}{8^{7/2}} \leq \frac{6\varepsilon^2}{8^{3/2}} < \varepsilon \text{ [will arrange]} \]
\[ \left[ \varepsilon < \frac{8^3}{6} \right] . \]
Claim 2:

\[(1-\delta)n\] \rightarrow \frac{(1-\delta)n}{n} \leq \delta \leq \frac{n}{g}

Proof: follows from tensor product + ensure decoding.

Proof of Claim 3:

Fraction of pts. on bad planes \( \leq \frac{18T}{\delta^2} \)

If \( f \neq g \in \mathbb{C}^n \times \mathbb{C}^n \) then pt is on bad plane

\[ f(i,j,k) = (A(i,j,k) = B(i,j,k) = \mathbb{P}(i,j,k)) \]

Fraction of such pts \( \leq \frac{T}{n} \).

\[ \Rightarrow \text{dist}(f, g) \leq \frac{T + 18T}{\delta^2} = \left(1 + \frac{18}{\delta^2}\right)T. \]

\[ \Rightarrow \text{Test is } \left(1 + \frac{18}{\delta^2}\right) - \text{robust} \]

in general some dependence on m.
Low degree Testing \([RS, ALMSS, \ldots 6115]\)

- \(R_m(m, d, q)\): \(m\)-var poly over \(\mathbb{F}_2\) of deg \(\leq d\).
- Test: Pick random 2-dim plane \(P\);
  Verify \(\deg(f|_P) \leq d\).
- Robustness:
  \[T(f) \leq \mathbb{E}_{P} \left[ S(f|_P, \mathbb{F}_2^{\leq d}[t_1, t_2]) \right] \]
  \[S(f) = S(f, \mathbb{F}_2^{\leq d}[x_1, \ldots, x_m])\]

Thm: \(\exists \alpha > 0\) s.t. \(\forall m, q, d\) s.t. \(1 - \frac{d}{q} \geq \frac{\alpha}{2}\)
the two-dim test is \(\alpha\)-robust

Lemma 1: Sufﬁces to prove theorem for \(m \leq 5\).

Lemma 2: Thm hold for \(m \leq 5\).
Proof Ideas for Lemmas 2 & 1

Lemma 2:
1. Say direction $P$ is good if planes of form $E_a + P$ are close to deg. d poly for most $a$.
   - Say $m = 3$
   - Suppose $XY$-plane, $XZ$-plane & $YZ$-plane are good for $f$
   - Then by Viderman, $f$ is close to some polynomial $g$ with $\deg_x g, \deg_y g, \deg_z g \leq d$
   - if $d \leq \frac{q}{3} (1 - \epsilon)$ then $\deg g < (1 - \epsilon) \frac{q}{2}$.
   - So why $g$ on random plane will be close $f$.
   - But why $g$ on random plane has $\deg g = \deg (\text{poly})$?
   - $\Rightarrow \deg g$ while $f$ on random plane is close to $\deg d$ poly

$\Rightarrow g$ has $\deg d$. $\Box$

$d > \frac{q}{2}$ non-trivial: Actually $d \geq \frac{q}{2}$ not true!

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$\frac{q}{2} > d \geq \frac{q}{2}$: Non-trivial
Lemma 1: [New going back to Blum Luby Rubinfeld, R + S]

- Key notion: det $g = \text{local-decoder}(f)$.
- Need to prove 1) $g$ close to $f$.
  2) $g$ is a deg. $d$ poly.

1) Easy: if $g = f$ at $x$, then $\Pr[\text{test through } x \text{ rejects}] 
\geq \frac{1}{2}.

[\text{Markov } \Rightarrow \Pr[f(x) + g(x)] \leq 2 \cdot \mathbb{E}[T(f)].]

2) $g$ is deg $d$?

Fix $x$; $\text{Decoder}^f(l_1, x) = \text{Decode } f|_{l_1}$ & output
value at $x$.

- Question: $\exists x \Pr[\text{Decoder}^f(l_1, x) = \text{Decoder}^f(l_2, x)]$?
  (Necessary, and morally also sufficient)

- Answer: Yes ... for following reason.
Pick $l_1, l_2$ randomly through $x$ & then a random
3-dim cube $C$ containing $l_1, l_2$. 
Typical $2^d$-plane $P$ in $C$ is random; $\Rightarrow$ close to deg $d$ poly.

By Lemma 2 $\Rightarrow$ fl$_c$ close to deg $d$ poly. $g$  
$\Rightarrow$ fl$_c$ close to & nearest poly to $fl_{l_1} = g|_{l_1}$  
& nearest poly $fl_{l_2} = g|_{l_2}$  
$\Rightarrow$ Decoder $(l_1, x) = Decoder^t (l_2, x) = g(x)$.

Can use this to prove $g|_{l} = g$ deg $d$ poly $\& c$.

$[\forall x, \forall l \exists x \ g(x) = \text{Decoder} [l(x)] ]$

$\forall x \ g(x) = c$, if $d < q^{1/2}$.

Asides on LTCs

Major ingredients in PCP theory:
- Tests of long code;
- Short long code;
- Hadamard code;
- Low-degree polynomials.

All central.