Invariance in Property Testing

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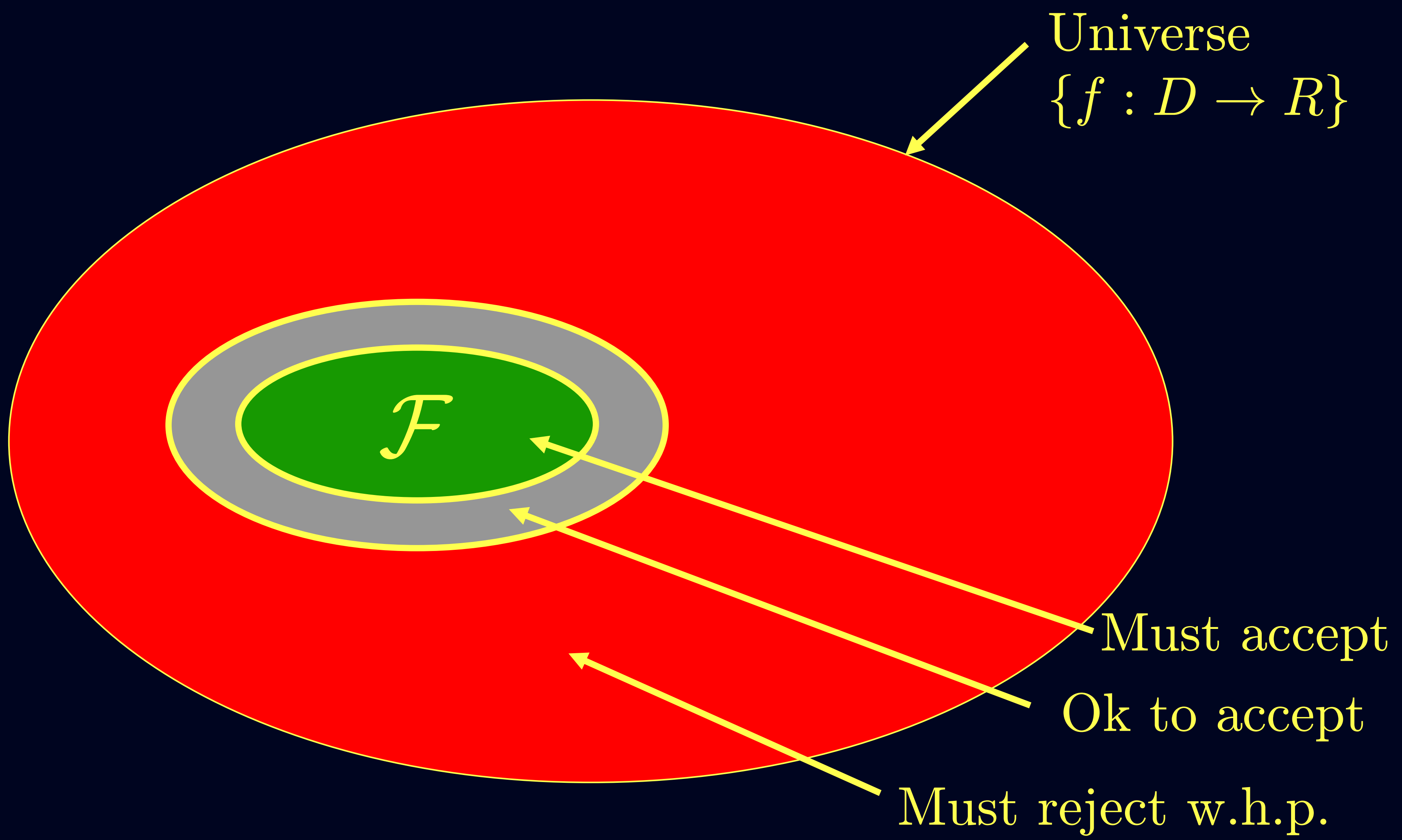
Property Testing

- Goal: "Efficiently" determine if some "data" "essentially" satisfies some given "property".
- Formalism:
  - Data: $f : D \to R$ given as oracle
    D finite, but huge. R finite, possibly small
  - Property: Given by $\mathcal{F} \subseteq \{f : D \to R\}$
  - Efficiently: $o(D)$ queries into f. Even $O(1)$!
  - Essentially: Must accept if $f \in \mathcal{F}$
    Ok to accept if $f \approx g \in \mathcal{F}$.

Property Testing

- Distance: $\delta(f,g) = \Pr_{x \in D}[f(x) \neq g(x)]$
  $\delta(f,\mathcal{F}) = \min_{g \in \mathcal{F}}\{\delta(f,g)\}$
  f $\approx_\epsilon g$ if $\delta(f,g) \leq \epsilon$.
- Definition:
  $\mathcal{F}$ is (q,α)-locally testable if
  $\exists$ a q-query tester that
  accepts $f \in \mathcal{F}$ with probability one
  rejects $f \notin \mathcal{F}$ with probability $\geq \alpha \cdot \delta(f,\mathcal{F})$.
- Notes: q-locally testable implies $\exists \alpha > 0$
  locally testable implies $\exists q = O(1)$
  Weaker testing: can reject $f \in \mathcal{F}$ with small prob.

Property Testing (Pictorially)



Example: Pre-election Polling

- Domain = Population
  Range = $\{0, 1\}$

- Property: $\mathcal{F}$ = functions with majority 1

- Essentially:
  Must reject w.h.p. if $\Pr_{x \in D}[f(x) = 1] \leq 1/2 - \epsilon$

- Efficiency? Can test weakly with $\tilde{O}(1/\epsilon^2)$ queries.
  Chernoff bounds.

Modern Day Example: Testing Linearity

- Domain = $\text{Vector space } \mathbb{F}_2^n$
  Range = $\text{Field } \mathbb{F}_2$

- Property: $\mathcal{F} = \text{linear functions}$
  $\text{i.e., } \{f(x) = \langle a, x \rangle | a \in \mathbb{F}_2^n\}$

- Theorem [Blum,Luby,Rubinfeld '89]:
  Linearity is 3-locally testable.

- Test: Pick $x, y \in \mathbb{F}_2^n$ uniformly.
  Accept iff $f(x) + f(y) = f(x + y)$

Property Testing: Abbreviated History

- Prehistoric: Statistical sampling
  - E.g., “Majority = 1?”
- Linearity Testing [BLR’90], Multilinearity Testing [Babai, Fortnow, Lund ’91].
- Graph/Combinatorial Property Testing [Goldreich, Goldwasser, Ron ’94].
  - E.g., Is a graph “close” to being 3-colorable.
- Algebraic Testing [GLRSW,RS,FS,AKKLR,KR,JPSZ]
  - Is multivariate function a polynomial (of bounded degree).
- Graph Testing [Alon-Shapira, AFNS, Borgs et al.]
  - Characterizes graph properties that are testable.

Quest for this talk

- What makes a property testable?
- In particular for algebraic properties:
  - Current understanding:
    - Low-degree multivariate functions are testable.
    - Different proofs for different cases.
      - Linear functions
      - Low-degree polynomials
      - Higher degree polynomials over $\mathbb{F}_2$
      - Higher degree polynomials over other fields

Necessary Conditions for Testability

- One-sided error and testability:
  - Suppose f is rejected by a k-query 1-sided tester. Suppose queried points are $x_1, \ldots, x_k \in D$. Let $(x_i) = \alpha_i$.
  - Then for every function $g \in \mathcal{F}$, $\langle g(x_1), \ldots, b(x_k) \rangle \neq \langle \alpha_1, \ldots, \alpha_k \rangle$.
- Constraint: $C = \langle x_1, \ldots, x_k \rangle; S \subsetneq R^k$
  g satisfies C if $\langle g(x_1), \ldots, g(x_k) \rangle \in S$
  $\mathcal{F}$ satisfies C if every $g \in \mathcal{F}$ satisfies C.
- Conclusion: Testability implies Constraints.

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Constraints, Characterizations, Testing

- Strong testing:

Every $f \notin \mathcal{F}$ rejected by some k-local constraint.
Set of k-local constraints characterize $\mathcal{F}$.
$\exists C_1, \ldots, C_m$ s.t. $f \in \mathcal{F} \Leftrightarrow f$ satisfies C_j for every j.

- Conclusion: Testability implies Local Characterizations.
- Example:

$f \subseteq \{\mathbb{F}_2^n \to \mathbb{F}_2\}$ is linear iff
for all $x, y \in \mathbb{F}_2^n$, f satisfies $C_{x,y}$ where
$C_{x,y} = \langle x, y, x+y \rangle; S = \{000, 011, 101, 110\}$.

Characterizations Sufficient?

- NO! [Ben-Sasson, Harsha, Raskhodnikova]
  - Random 3-locally characterized error-correcting codes ("Expander Codes") are not o(D)-locally testable.
    - Property:

$$D = [n];\ R = \{0,1\};$$
$$\mathcal{F} = \text{set of functions that satisfy some random 3-ary } \mathbb{F}_2\text{-linear constraints.}$$

- Criticism: Random constraints too "asymmetric".
- Perhaps should consider more "symmetric" properties.

Invariance & Property testing

- Invariances (Automorphism groups):

  For permutation $\pi : D \to D$, $\mathcal{F}$ is π-invariant if
  $f \in \mathcal{F}$ implies $f \circ \pi \in \mathcal{F}$.

  $\mathrm{Aut}(\mathcal{F}) = \{\pi \mid \mathcal{F} \text{ is } \pi\text{-invariant}\}$

  Forms group under composition.

- Hope: If Automorphism group is “large” (“nice”), then property is testable.

Examples

- Majority:
  - Aut group $= S_D$ (full group).
  - Easy Fact: If $\mathrm{Aut}(\mathcal{F}) = S_D$ then $\mathcal{F}$ is $\mathrm{poly}(R, 1/\epsilon)$-locally testable.
- Graph Properties:
  - Aut. group given by renaming of vertices
  - [AFNS, Borgs et al.] implies *regular* graph properties testable.
- Matrix Properties: Have lots of symmetries – do they suffice?
- Algebraic Properties: What symmetries do they have? Will focus on this today.

Algebraic Properties & Invariances

- Properties:
  $D = \mathbb{F}^n$, $R = \mathbb{F}$ (Linearity, Low-degree, Reed-Muller)
  Or $D = \mathbb{K} \supseteq \mathbb{F}$, $R = \mathbb{F}$ (Dual-BCH) $(\mathbb{K}, \mathbb{F}$ finite fields$)$
- Automorphism groups?
  Linear transformations of domain.
  $\pi(x) = Ax$ where $A \in \mathbb{F}^{n \times n}$ (Linear-Invariant)
- Additional restriction: Linearity
  $f, g \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{F}$ implies $\alpha f + \beta g \in \mathcal{F}$
- Question: Are Linear, Linear-Invariant, Locally Characterized Properties Testable?

Linear-Invariance & Testability

- Question: Are Linear, Linear-Invariant, Locally Characterized Properties Testable?
  - Why?
    - Unifies previous results on Prop. Testing.
    - (Will show it also is non-trivial extension)
    - Nice family of 2-transitive group of symmetries.
    - Conjecture [Alon, Kaufman, Krivelevich, Litsyn, Ron] : Linear code with k-local constraint and 2-transitive group of symmetries must be testable.

Our Results

- Theorem 1: $\mathcal{F} \subseteq \{\mathbb{K}^n \to \mathbb{F}\}$ linear, linear-invariant, k-locally characterized implies $\mathcal{F}$ is $f(\mathbb{K}, k)$-locally testable.
- Theorem 2: $\mathcal{F} \subseteq \{\mathbb{K}^n \to \mathbb{F}\}$ linear, *affine*-invariant, has k-local *constraint* implies $\mathcal{F}$ is $f(\mathbb{K}, k)$-locally testable.

- Other stuff: Study of Linear-invariant Properties.

Linear Invariant Properties

Examples of Linear-Invariant Families

– Polynomials in $\mathbb{F}[x_1, \ldots, x_n]$ of degree at most d

– Traces of Poly in $\mathbb{K}[x_1, \ldots, x_n]$ of degree at most d

– (Traces of) Homogenous polynomials of degree d

– $\mathcal{F}_1 + \mathcal{F}_2$, where $\mathcal{F}_1$, $\mathcal{F}_2$ are linear-invariant.
Polynomials supported by degree $2, 3, 5, 7$ monomials.

What Dictates Locality of Characterizations?

– Precise locality not yet understood:
  Depends on p-ary representation of degrees.
  Example: $\mathcal{F}$ supported by monomials $x^{p^i+p^j}$
  behaves like degree two polynomial

– For affine-invariant family dictated (coarsely)
  by highest degree monomial in family

– For some linear-invariant families,
  can be *much* less than the highest degree monomial.

Example: $\mathbb{K} = \mathbb{F} = \mathbb{F}_7$; $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$
  $\mathcal{F}_1$ = poly of degree at most 16
  $\mathcal{F}_2$ = poly supported on monomials of degree $3 \mod 6$.
  $\text{Degree}(\mathcal{F}) = \Omega(n)$; $\text{Locality}(\mathcal{F}) \leq 49$.

Analysis Ingredients

- Monomial Extraction:

  E.g., $xy^2 + xyz + x^4 \in \mathcal{F}$ implies $xyz \in \mathcal{F}$

- Monomial Spread:

  $x^5 \in \mathcal{F}$ implies $x^4 y, x^3 y^2$ also in $\mathcal{F}$ (if char($\mathbb{F}$) large)

Suffices for affine-invariant families.
For linear-invariant families, need to define
the right parameter and bound locality
weakly in terms of it.

Local Testing

Key Notion: Formal Characterization

– $\mathcal{F}$ is formally characterized if
$\exists$ a *single* constraint $C = (\langle x_1, \ldots, x_k \rangle, S)$ such that
$\{C \circ \pi\}_{\pi \in \mathrm{Aut}(\mathcal{F})}$ characterize $\mathcal{F}$.

Theorem: If $\mathcal{F}$ is formally characterized by
a k-local constraint (with some restrictions)
then it is k-locally testable.

BLR (and our) analysis

BLR Analysis: Outline

- Have f s.t. $\Pr_{x,y}[f(x)+f(y) \neq f(x+y)] = \delta < 1/20$.
  Want to show f close to some $g \in \mathcal{F}$.
- Define $g(x) = \text{most likely}_y\{f(x+y) - f(y)\}$.
- If f close to $\mathcal{F}$ then g will be in $\mathcal{F}$ and close to f.
- But if f not close? g may not even be uniquely defined!
- Steps:
  - Step 0: Prove f close to g
  - Step 1: Prove *most likely* is overwhelming majority.
  - Step 2: Prove that g is in $\mathcal{F}$.

BLR Analysis: Step 0

- Define $g(x) =$ most likely $_y\{f(x+y) - f(y)\}$.

Claim: $\Pr_x[f(x) \neq g(x)] \leq 2\delta$

- Let $B = \{x | \Pr_y[f(x) \neq f(x+y)f(y)] \geq \frac{1}{2}\}$
- $\Pr_{x,y}[\text{linearity test rejects } | x \in B] \geq \frac{1}{2}$

  $\Rightarrow \Pr_x[x \in B] \leq 2\delta$
- If $x \notin B$ then $f(x) = g(x)$

BLR Analysis: Step 1

$\mathrm{Vote}_x(y)$

- Define $g(x) = \text{most likely}\ {}_y\{f(x+y) - f(y)\}$.
- Suppose for some x, $\exists$ two equally likely values.
  Presumably, only one leads to linear x, so which one?
- If we wish to show g linear,
  then need to rule out this case.

Lemma: $\forall\, x,\ \Pr_{y,z}[\mathrm{Vote}_x(y) \neq \mathrm{Vote}_x(z))] \leq 4\delta$

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?	$f(y)$	$-f(x+y)$
$f(z)$	$f(y+z)$	$-f(y+2z)$
$-f(x+z)$	$-f(2y+z)$	$f(x+2y+2z)$

Prob. Row/column sum non-zero $\leq \delta$.

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BLR Analysis: Step 2 (Similar)

Lemma: If $\delta < \frac{1}{20}$, then $\forall\, x, y$, $g(x) + g(y) = g(x+y)$

$g(x)$	$g(y)$	$-g(x+y)$
$f(z)$	$f(y+z)$	$-f(y+2z)$
$-f(x+z)$	$-f(2y+z)$	$f(x+2y+2z)$

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Our Analysis: Outline

- f s.t. $\Pr_L[\langle f(L(x_1), \ldots, f(L(x_k))\rangle \in V] = \delta \ll 1$.
- Define $g(x) = \alpha$ that maximizes
  $\Pr_{\{L | L(x_1) = x\}}[\langle \alpha, f(L(x_2)), \ldots, f(L(x_k))\rangle \in V]$

- Steps:
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Same as before

- Steps:
  - Step 0: Prove f close to g
  - Step 1: Prove "most likely" is overwhelming majority.
  - Step 2: Prove that g is in $\mathcal{F}$.

$\mathrm{Vote}_x(L)$

Matrix Magic?

- Define $g(x) = \alpha$ that maximizes

$$\Pr_{\{L \mid L(x_1)=x\}}[\langle \alpha, f(L(x_2)), \ldots, f(L(x_k))\rangle \in V]$$

Lemma: $\forall\, x,\ \Pr_{L,K}[\mathrm{Vote}_x(L) \neq \mathrm{Vote}_x(K))] \leq 2(k-1)\delta$

x	$L(x_2)$	$\cdots$	$L(x_k)$
$K(x_2)$			
$\vdots$		?	
$K(x_k)$			

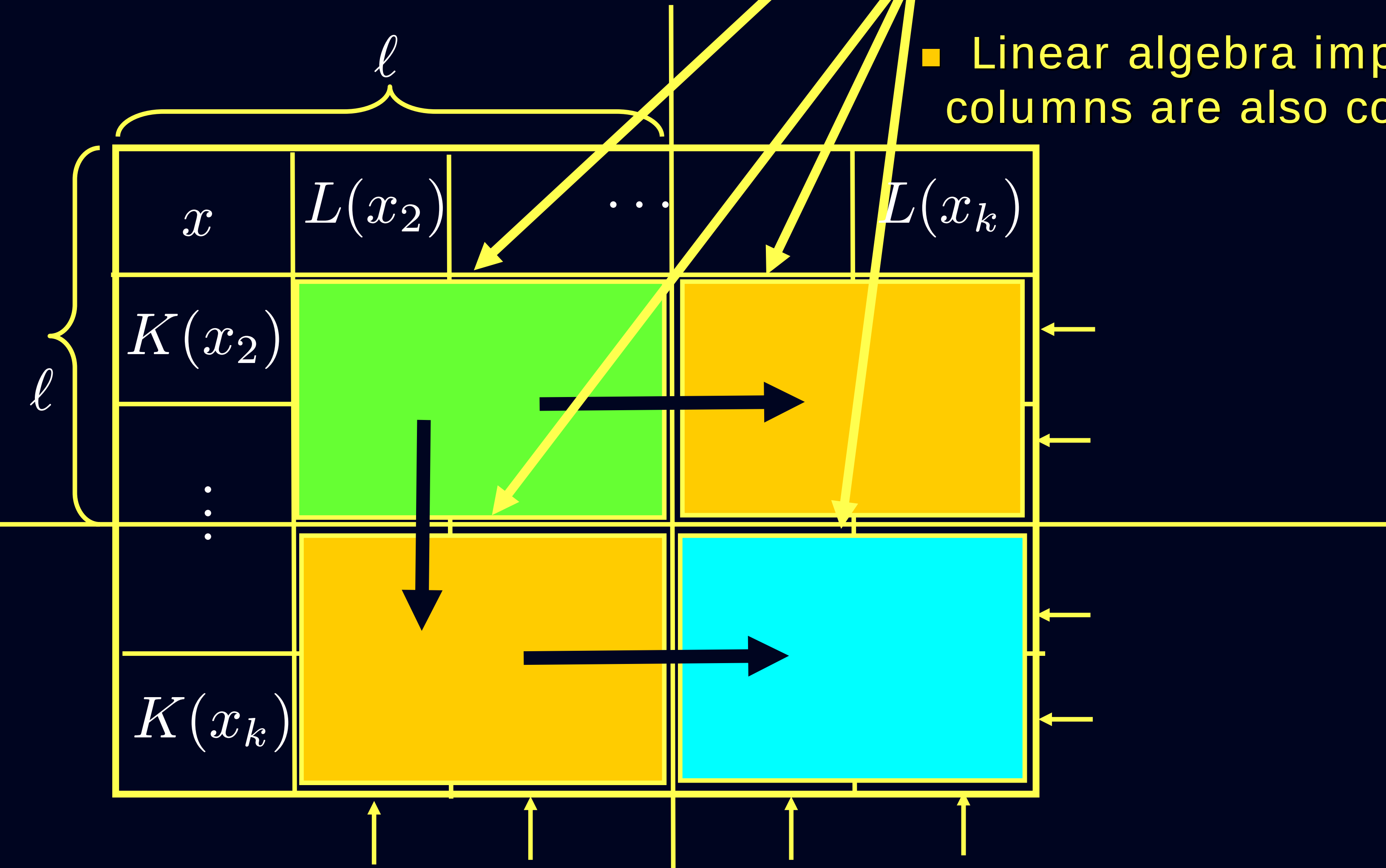
Matrix Magic?

x	$L(x_2)$	$\cdots$	$L(x_k)$
$K(x_2)$			
$\vdots$		?	
$K(x_k)$			

- Want marked rows to be random constraints.
- Suppose $x_1, \ldots, x_\ell$ linearly independent; and rest dependent on them.

Matrix Magic?

- Fill with random entries
- Fill so as to form constraints
- Linear algebra implies final columns are also constraints.



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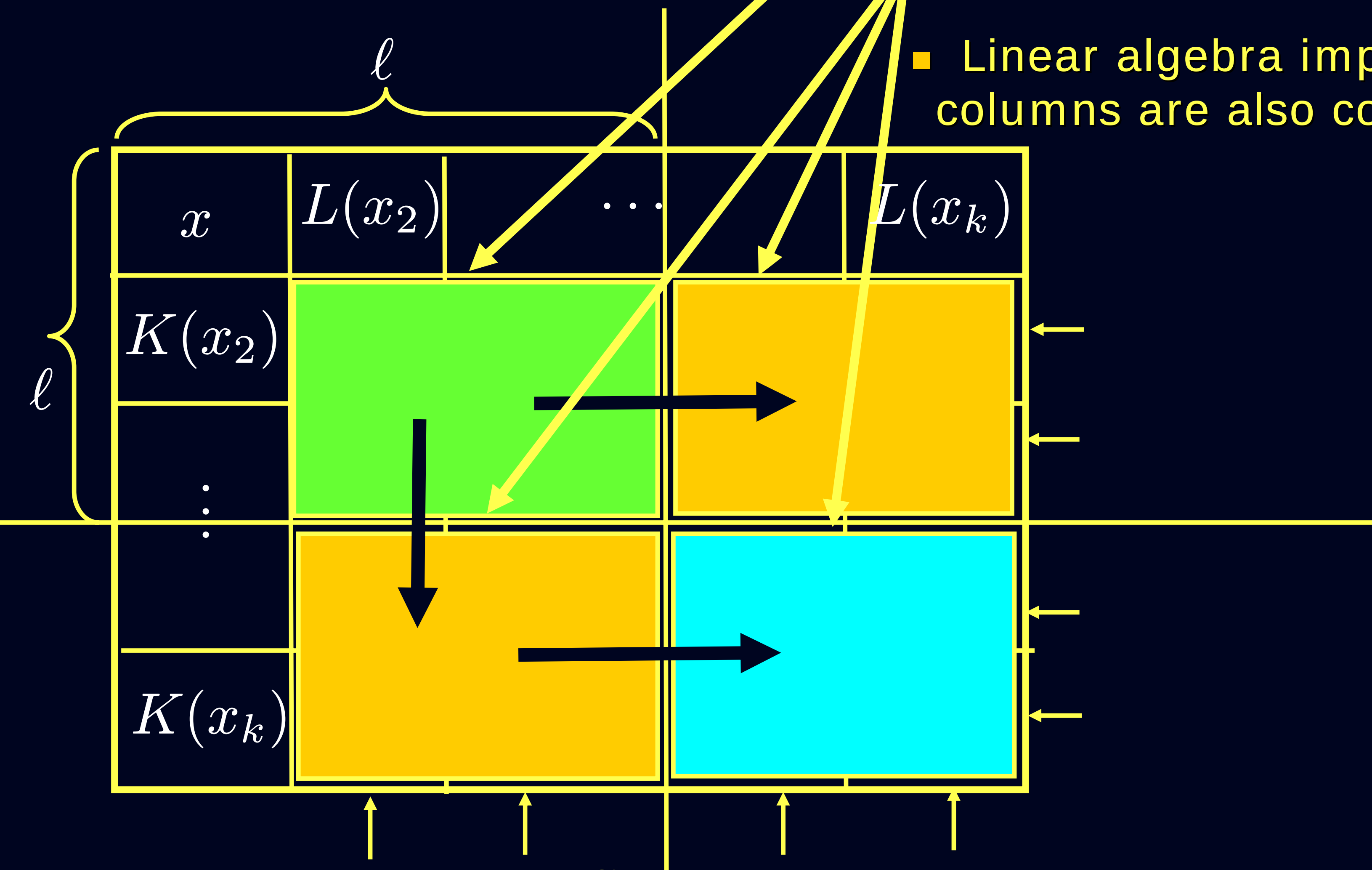

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Conclusions

- Linear/Affine-invariant properties testable if they have local constraints.
- Gives clean generalization of linearity and low-degree tests.
- Future work: What kind of invariances lead to testability (from characterizations)?