Local Testability and Decodability of Sparse Linear Codes

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Local (Sublinear-time) Algorithmics

- Data getting ever-larger
  - Need algorithms that can infer "global" properties from "local" observations ...
- Led to
  - Property testing, Sublinear-time algorithms
- Common themes:
  - Oracle-access to input, implicit output.
  - Answers of the form: "input close to having property"

Error-Correcting Codes

- Code: $C \subseteq \{0,1\}^n$ image of $E : \{0,1\}^k \to \{0,1\}^n$
- Distance ...
  - ... between sequences: $\delta(x,y) = \Pr_i[x_i \neq y_i]$
  - ... of code: $\delta(C) = \min_{x \neq y \in C}\{\delta(x,y)\}$
- Algorithmic Problems:
  - Encode: Compute E
  - Detect Errors: Given $r \in \{0,1\}^n$, is $r \in C$?
    Or $\exists x \in C$ s.t. $\delta(r,x) \leq \epsilon$?
  - Decode: Given $r \in \{0,1\}^n$ s.t. $\exists x \in C$
    with $\delta(r,x) \leq \epsilon$, compute x.

Local Algorithmics in Coding

- Encoding: Can not be performed “locally”
  - Single bit change in input should alter constant fraction of output!

- Testing, Decoding, Error-correcting … can be performed locally. Furthermore
  - They are very natural problems.
  - Have many applications in theory (PCP, PIR, Hardness amplification).
  - Lots of interesting effects are achievable.

Local Algorithmic Problems

- Common framework: Fixed code $C \in \{0,1\}^n$; Oracle access to $r \in \{0,1\}^n$; Only k queries allowed.
- Local Testing: accept if $r \in C$
  reject (with $\Omega(1)$ prob.) if $\delta(r, C) \geq \epsilon$.
- Local Self-Correction:
  Promise: $\exists c \in C$ s.t. $\delta(c, r) \leq \epsilon$.
  Given $i \in [n]$, compute c_i
- Local Decoding:
  Setup: Fix $E : \{0,1\}^k \to \{0,1\}^n$ s.t. $C = \mathrm{Image}(E)$.
  Promise: $\exists m$ s.t. $\delta(E(m), r) \leq \epsilon$.
  Given $i \in [k]$, compute m_i

Example: Hadamard Codes

- **Encoding:** Given $m \in \{0,1\}^{\log n}$, and $x \in \{0,1\}^{\log n}$
  $$E(m)_x = \sum_{i=1}^{\log n} m_i x_i \pmod 2$$
- **Test:** Accept iff $r_x + r_y = r_{x+y}$
- **Correction:** Given $x \in \{0,1\}^{\log n}$, pick $y \in \{0,1\}^{\log n}$ uniformly and output $r_{x+y} - r_y$
- **Decoding:**

  ith bit of message is e_ith coordinate of its encoding.

Brief History

- Local Decoding/Self-Correcting:
  - [Beaver-Feigenbaum], [Lipton], [Blum-Luby-Rubinfeld] – instances of Local Decodability.
  - [Katz-Trevisan] – first definition.
  - ...
- Locally Testable Codes:
  - [Blum-Luby-Rubinfeld], [Babai-Fortnow-Lund] – first instances.
  - [Arora], [Rubinfeld-Sudan], [Spielman], [Goldreich-Sudan] – definitions.
  - ...

Constructions of Locally X-able Codes

- Basic codes: Algebraic in nature.
  - Analysis:
    - Decoding: typically simple, uses algebra.
    - Testing: more complex.

- Better codes: Careful compositions of basic codes.
  - Exception: [Meir ’08] – not algebraic.

- Questions:
  - Do we need all this algebra/careful constructions?
  - Can we derive local algorithms from “classical” parameters?
  - Can randomly chosen codes have local algorithms?

Our Results

- Theorem (Informal): Every "sparse", "linear" code of "large distance" is locally testable, correctible.

- Linear? C linear if $x, y \in C \Rightarrow x + y \in C$
- Sparse? C is t-sparse if $|C| \leq n^t$
- Large Distance?
  C has γ-large-distance if $\delta(C) \geq \frac{1}{2} - n^{-\gamma}$

Theorem 1: $\forall \gamma > 0$, $t < \infty$, $\exists k < \infty$ such that if C is t-sparse, linear and has γ-large-distance then C is k-locally testable.

Our Results (contd.)

- Linear? C linear if $x, y \in C \Rightarrow x + y \in C$
- Sparse? C is t-sparse if $|C| \leq n^t$
- Large Distance?

  C has γ-large-distance if $\delta(C) \geq \frac{1}{2} - n^{-\gamma}$
- Balanced?

  C is γ-balanced if $\forall x \neq y \in C$,

  $$\frac{1}{2} - n^{-\gamma} \leq \delta(x, y) \leq \frac{1}{2} + n^{-\gamma}.$$

Theorem 2: $\forall \gamma > 0$, $t < \infty$, $\exists k < \infty$ such that if C is t-sparse, linear and is γ-balanced then C is k-locally correctible.

Corollaries

- Reproduce old results: Hadamard, dual-BCH
- New codes:
  - Random sparse linear codes (decodable under *any* linear encoding).
  - dual-BCH variants
    $$\{\mathrm{Trace}(c_1 x^{i_1} + \cdots c_t x^{i_t}) | c_1, \ldots, c_t \in \mathbb{F}_{2^{\log n}}\},$$
    $$i_1, \ldots, i_t < \sqrt{n}$$
- Nice closure properties: (Subcodes, Addition of new coordinates, removal of few coordinates)

Previously …

- [Kaufman-Litsyn] Similar result + techniques. Main differences:
  - Required $\gamma \geq \frac{1}{2}$. So $\delta(C) \geq \frac{1}{2} - \frac{1}{\sqrt{n}}$
  - Worked only for balanced codes.
  - Only proved local testability … no correctibility

Proof Techniques

- Modifying (simplifying? extending?) the proofs of [Kaufman Litsyn '05] (some ideas go back to [Kiwi 95]).

- Buzzwords: Duality, MacWilliams Identities, Krawtchouk Polynomials, Johnson bounds.

Linearity, Duality, & Testing

- Dual of a Code:

$$C^{\perp} = \{y \in \{0,1\}^n | \langle x, y\rangle \triangleq \bigoplus_{i=1}^{n} x_i y_i = 0, \forall x \in C\}$$

- Canonical (only) test for membership in C:

Pick low-weight $y \in C^{\perp}$

Test $\langle r, y\rangle = \bigoplus_{i \in 1_y} r_i = 0$

$\mathrm{wt}(y) = k \Rightarrow$ Test is k-local

$$1_y \triangleq \{i | y_i = 1\}$$

$$\mathrm{wt}(y) \triangleq |1_y|$$

- Canonical self-corrector:

To compute c_i, pick low-weight y s.t. $y_i = 1$

output $\bigoplus_{j \in 1_y - \{i\}} r_j$

Questions:

- Does $C^\perp$ even have any low-weight codewords?
- Is the distribution of non-zero coords. of low-weight y s.t. $y_i = 1$ roughly uniform?
- How to even analyze the test?

Path to answers

- Need "weight distribution" of some codes:
  Weight distribution: $C_0, \ldots, C_n$, where
  $$C_i = \# \text{ codewords in } C \text{ of weight } i.$$
- Testing + Correcting: Weight distribution of $C^\perp$
  Specifically $C_k^\perp$
- Testing: [Kiwi, KL]
  Also need weight distribution of $(C \cup (C + r))^\perp$.
  Specifically, $(C \cup (C + r))_k^\perp$
- Correcting: [New]
  Wt. distribution of $C^{-i}, C^{-\{i,j\}}$
  $(C^{-i} : C$ with ith coordinate deleted.)

Dual Weight Distribution?

- MacWilliams Identities: Can compute weight distribution of dual from weight distribution of primal … exactly!

- Don’t have primal distribution exactly … Can coarse information suffice?
  - [Kiwi] - Manages to compute primal info. exactly.
  - [Kaufman-Litsyn] – Find out a lot about primal distribution.
  - [Our hope] – Less precise info. sufficient.

MacWilliams Identities: Precise Form

- Krawtchouk Polynomials

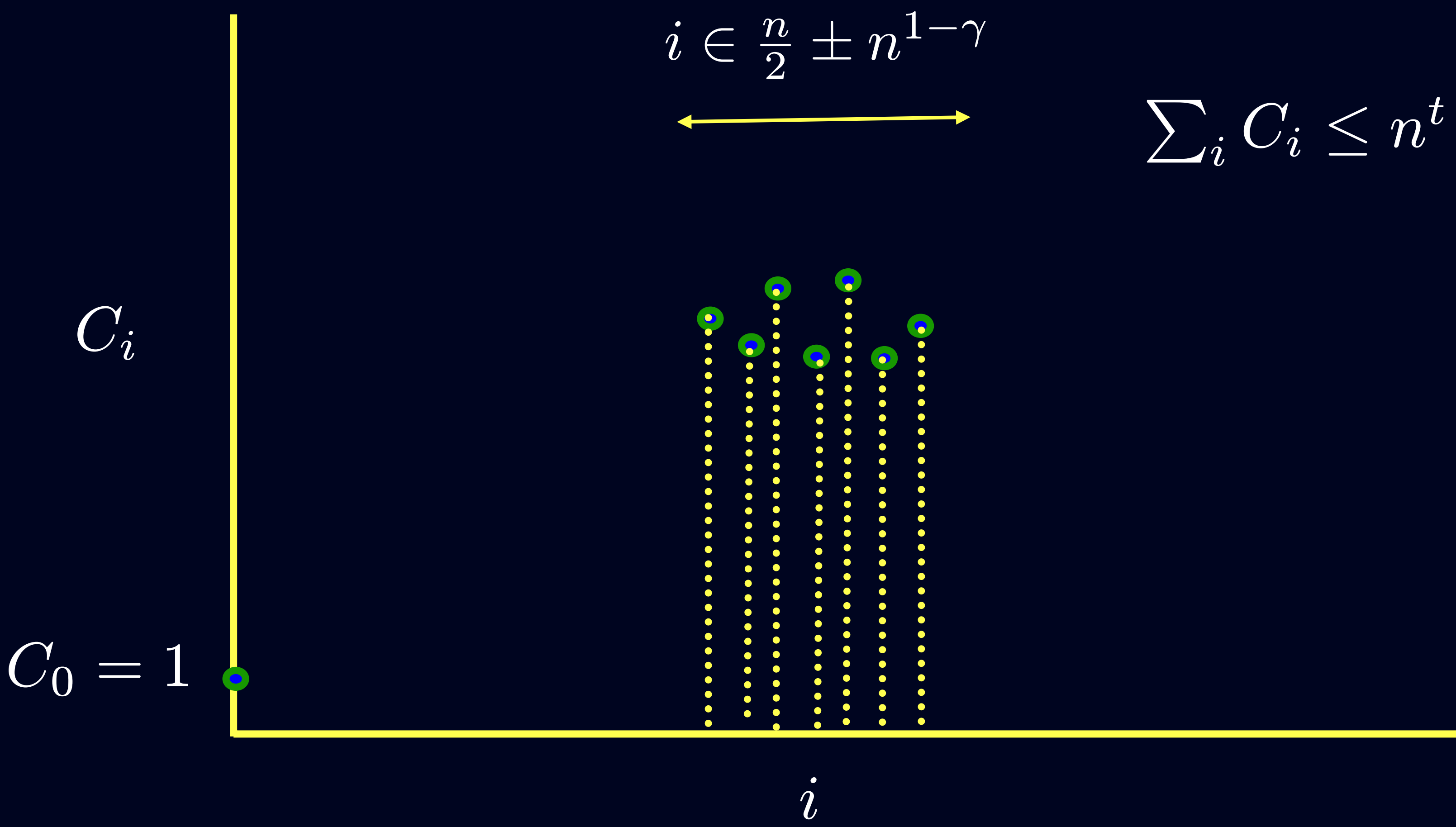
$$P_k(i) = \sum_{j=0}^{k} (-1)^j \binom{i}{j} \binom{n-i}{k-j}$$

- Dual Weight Distribution

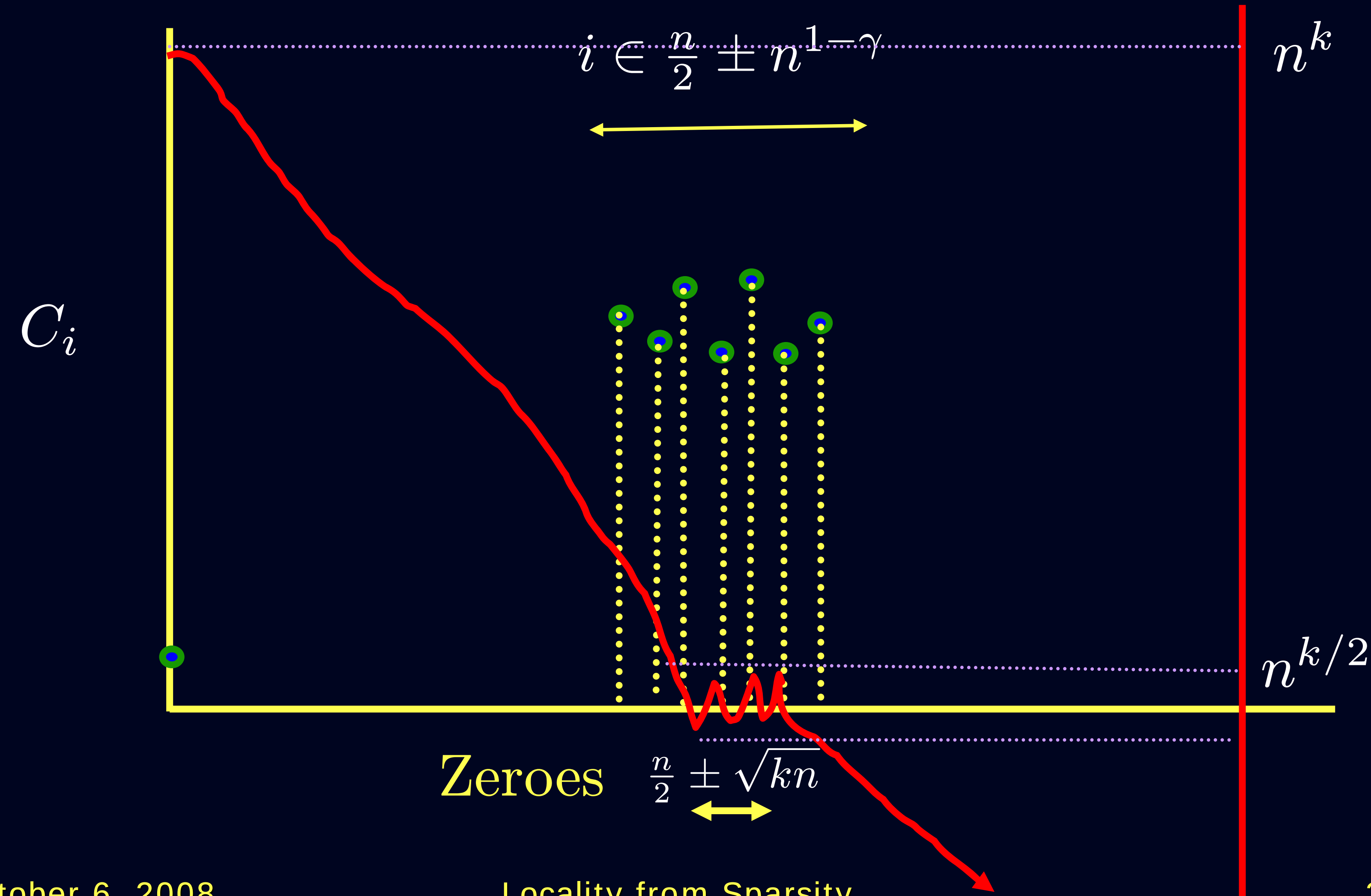
$$C_k^{\perp} = \frac{1}{|C|} \cdot \sum_{i=0}^{n} P_k(i) C_i$$

- Double summation! Many negative terms. Cancellations?

Primal Weight Distribution (Balanced)



Krawtchouk Polynomial (k odd)



Krawtchouk Polynomial (k odd)

Low-weight codewords in dual

- Can conclude: constant weight codewords exist.

$$C_k^\perp \approx \frac{1}{|C|} \cdot \binom{n}{k} \cdot (1 \pm n^{t-\gamma k})$$

- Very tight bound $(\text{If } k \gg t/\gamma)$
- Leads to self-corrector

Analysis of self-corrector

- Need to understand
$$C_{k,i}^{\perp} = |\{y \in C^{\perp} | \mathrm{wt}(y) = k \text{ and } \mathrm{y_i} = 1\}|$$

- New Code: $C^{-i} = C$ with ith coordinate deleted.
$$= \{\pi(y) | y \in C\}.$$

- Claim: $(C^{-i})^{\perp} = \{\pi(y) | y \in C^{\perp} \text{ s.t. } y_i = 0\}$
and so $C_{k,i}^{\perp} = C_k^{\perp} - (C^{-i})_k^{\perp}$

- But C^{-i} is sparse and balanced
and so can determine $(C^{-i})_k^{\perp}$

Analysis of self-corrector (contd.)

- Plugging in bounds:

$$\Pr_{y \in C_k^\perp}[y_i = 1] \approx k/n(1 \pm n^{-c})$$

- Similar calculations with $C^{-i,j}$ yield:

$$\text{Events } y_i = 1 \text{ and } y_j = 1 \text{ roughly independent} \\ \text{if } y \leftarrow C_k^\perp.$$

- Conclude: Self-corrector computes c_i correctly w.p. $\geq 1 - O(\epsilon \cdot t/\gamma)$ from ϵ-corrupted received word.

Analysis of Tester (balanced case)

- Need to analyze $\mathrm{span}(C,r)_k^{\perp}$

  where $\mathrm{span}(C,r) = C \cup (C+r)$
- Specifically, want: $\Pr_{y \in C_k^{\perp}}[y \notin \mathrm{span}(C,r)_k^{\perp}] = \Omega(\epsilon)$.

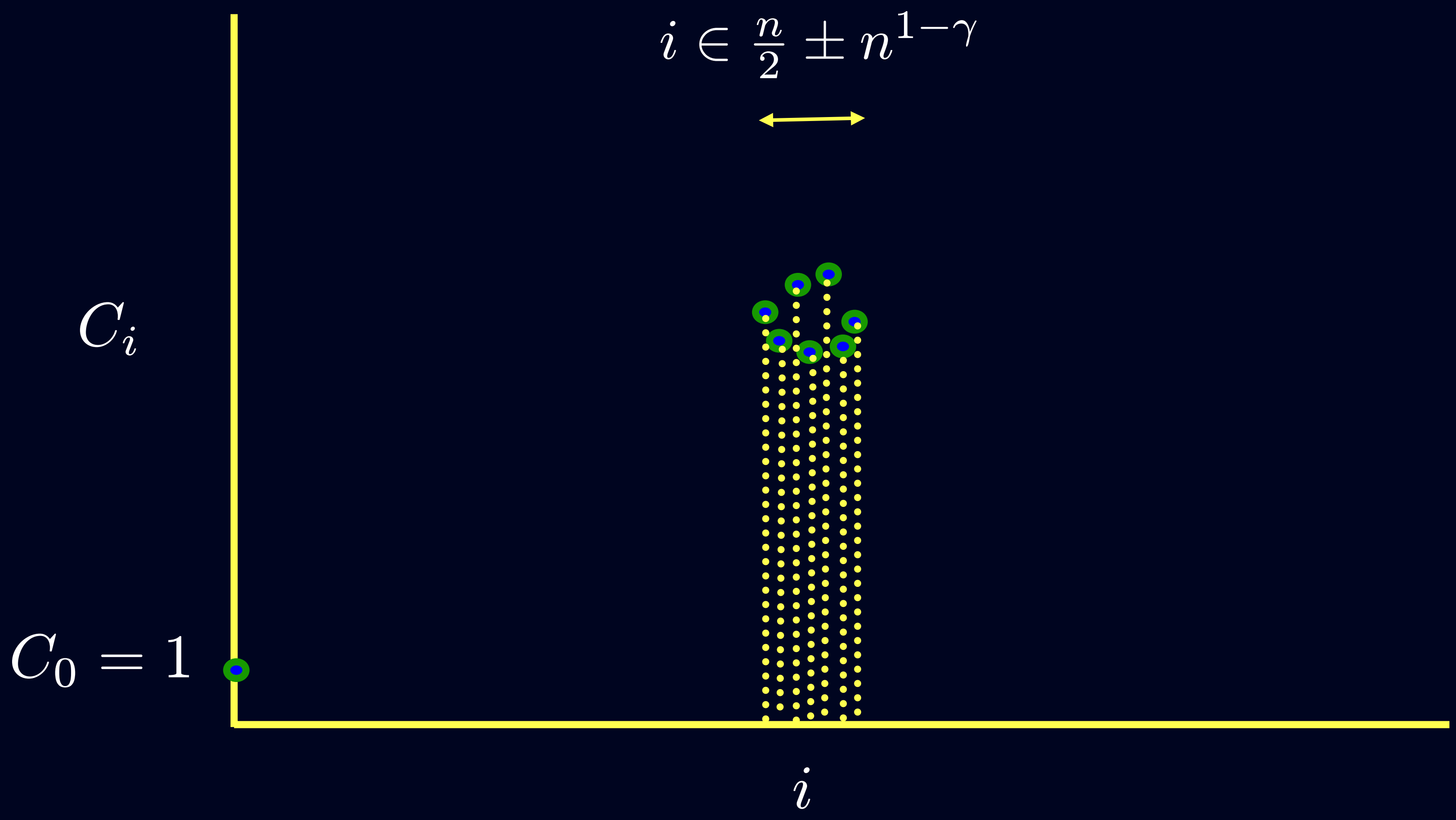
  $\Leftrightarrow \ \mathrm{span}(C,r)_k^{\perp} \leq (1-\Omega(\epsilon)) \cdot C_k^{\perp}$

- Easy fact (from MacWilliams Identities)

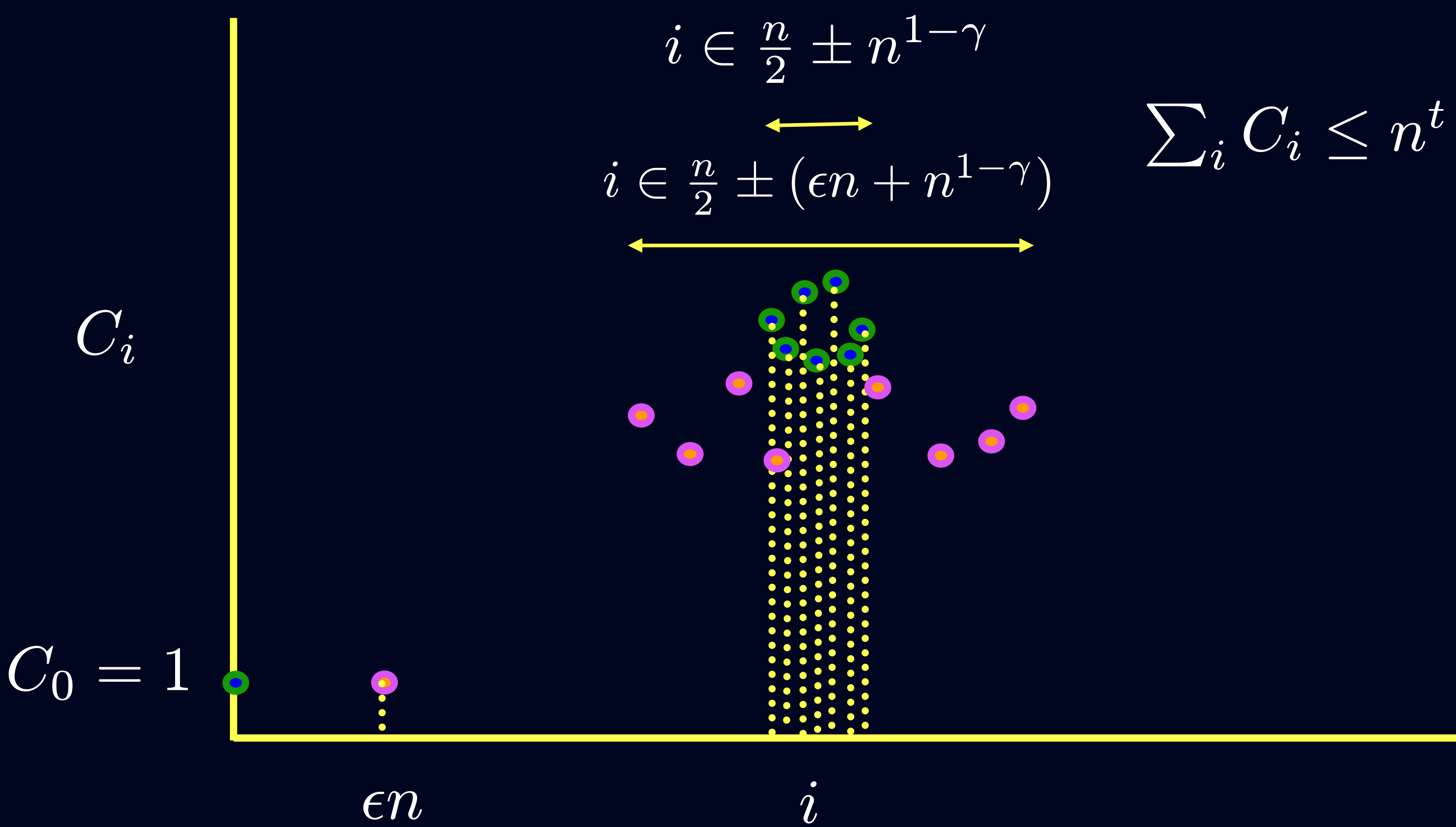
  $\mathrm{span}(C,r)_k^{\perp} = \frac{1}{2} \cdot C_k^{\perp} + \frac{1}{2} \cdot \frac{1}{|C|} \cdot \sum_{i=0}^{n} P_k(i) \cdot (C+r)_i$

- Suffices to analyze second term. But what does the weight distribution of $C+r$ look like? and how does $P_k(\cdot)$ interact with this?

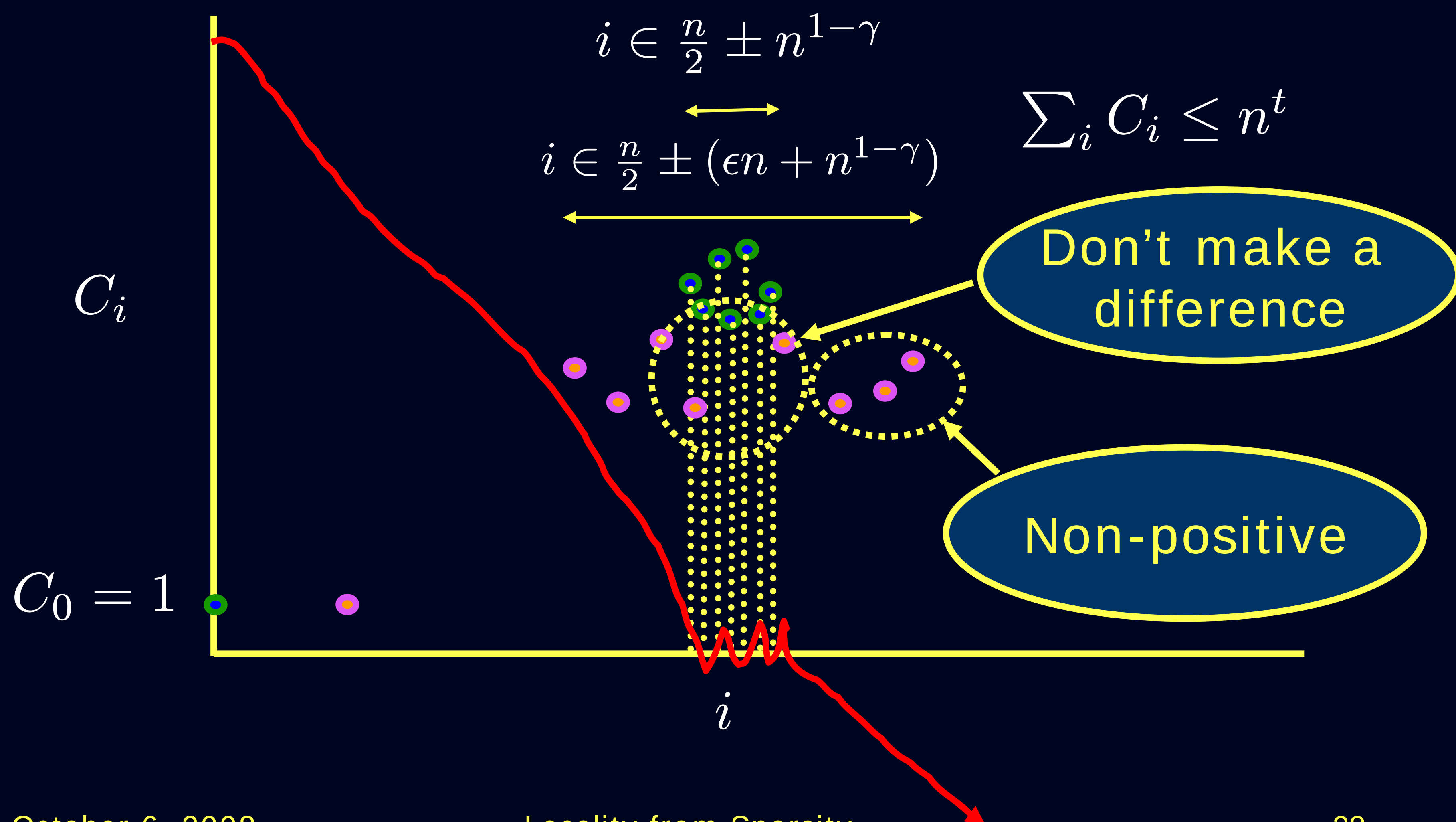
Weight Distribution of C+r (vs. C)



Weight Distribution of C+r (vs. C)



Inner Product with Krawtchouk's



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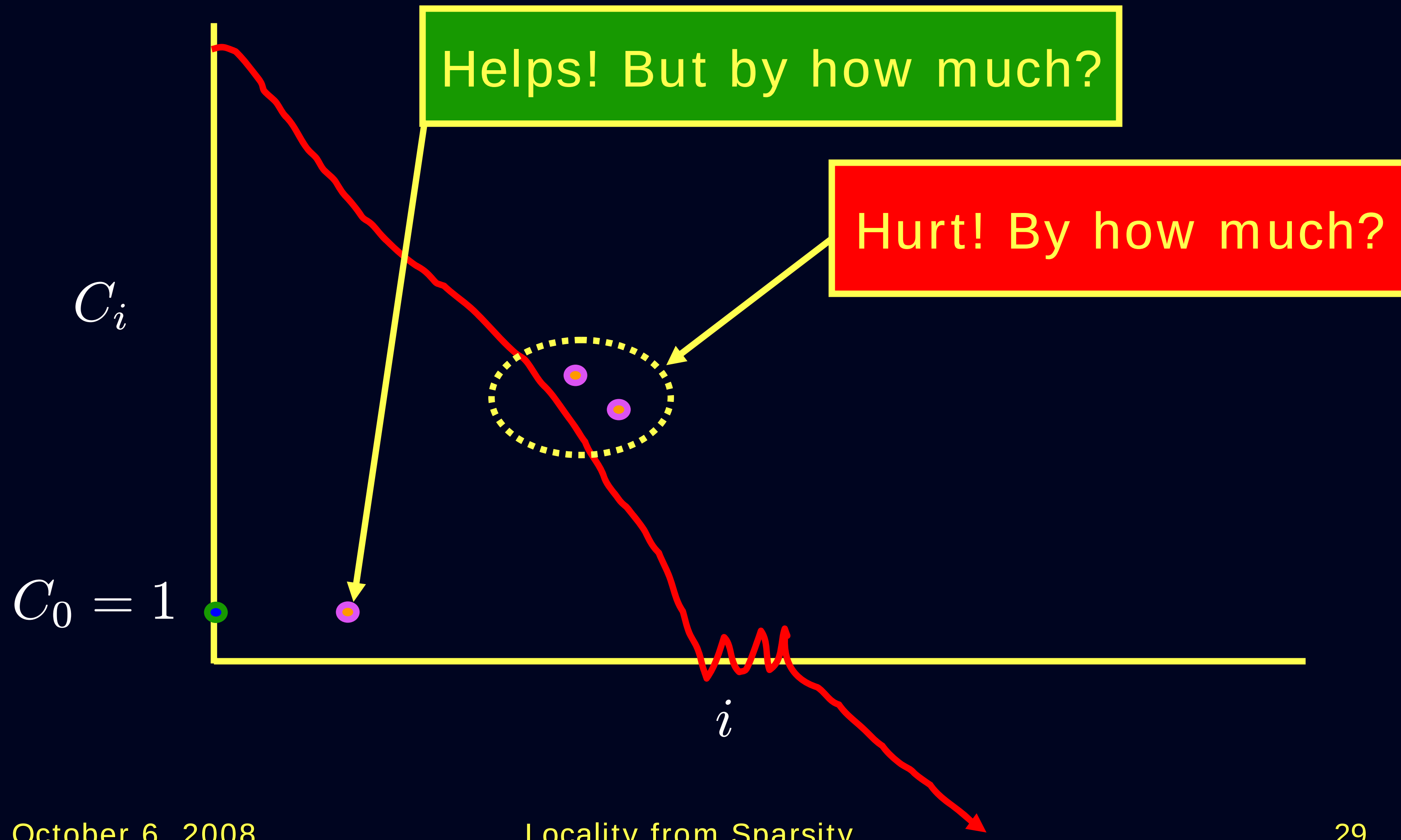

More Bounds

- Some weak Krawtchouk bounds:

  1. $P_k(\epsilon n) \leq (1-\epsilon)P_k(0)$ (the "helpful" part)

  2. $P_k(i) \leq (n-2i)^k/k!$ (For i in our range. Useful to limit the "hurt")

- Bound 2. not sufficient to bound the "hurt" ... but can combine with "Johnson bound"

- Johnson Bound:

  Code of relative distance $1/2 - \tau$ can not have too many codewords in ball of radius $1/2 - \sqrt{\tau}$

Putting all the bounds together

- Can conclude:

$$\frac{1}{|C|} \cdot \sum_{i=0}^{n} P_k(i)(C+r)_i \leq (1-\Omega(\epsilon)) \cdot C_k^{\perp}$$

- Implies test rejects ϵ-corrupted codeword with probability $\Omega(\epsilon)$.

Unbalanced codes?

- Many things break down ...
- E.g., If $\overline{1} \in C$ then $C_k^{\perp} = 0$ for odd k.
- Our approach:
  - Step 1: Codes of max. wt. $\leq 5/8n$ (weakly balanced).
  - Step 2: Reduce general case to weakly balanced case.

Weakly balanced codes

- Can now prove $C_k^\perp > 0$ for odd k.
- But can't get a precise bound on $C_k^\perp$.
- Instead, we bound $C_k^\perp - (\text{span}(C, r))_k^\perp$ directly;
  - Show that contribution of any word to both terms is roughly the same (Uses some properties of $P_{k-1}(\cdot)$.)
  - Show that contribution of the coset leader drops by $\Omega(\epsilon)$-factor.

Reducing general codes to w.b. codes

- Write $C = \tilde{C} + \mathrm{span}(x, y, z)$ where $\tilde{C}$ is weakly-balanced.
- Test if $\exists\ u \in \mathrm{span}(x, y, z)$ such that $r + u \in \tilde{C}$.
- Yields tester for all binary, linear, sparse, high-distance codes.

Conclusions / Questions

- Simpler proof for random codes? (Some work by Shachar Lovett, Or Meir)
- Self-correct imbalanced codes?
- Are random sparse codes locally list-decodable?
- Is this just a logarithmic saving in locality?
- Are there other ways to pick broad classes of testable codes (at “random”)?