Invariance in Property Testing

Madhu Sudan
Microsoft/MIT
Property Testing

- ... of functions from $D$ to $R$:
  - Property $P \subseteq \{D \to R\}$

- Distance
  - $\delta(f, g) = \Pr_{x \in D} [f(x) \neq g(x)]$
  - $\delta(f, P) = \min_{g \in P} [\delta(f, g)]$
  - $f$ is $\varepsilon$-close to $g$ ($f \approx_\varepsilon g$) iff $\delta(f, g) \leq \varepsilon$.

- Local testability:
  - $P$ is $(k, \varepsilon, \delta)$-locally testable if $\exists k$-query test $T$
    - $f \in P \Rightarrow T_f$ accepts w.p. $1-\varepsilon$.
    - $\delta(f, P) > \delta \Rightarrow T_f$ accepts w.p. $\varepsilon$.

- Notes: want $k(\varepsilon, \delta) = O(1)$ for $\varepsilon, \delta = \Omega(1)$. 
Brief History

- [Blum,Luby,Rubinfeld – S’90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F’90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing
- [Goldreich,Goldwasser,Ron]
  - Graph property testing
- Since then ... many developments
  - Graph properties
  - Statistical properties
  - ...
  - More algebraic properties
Specific Directions in Algebraic P.T.

- **More Properties**
  - **Low-degree** \((d < q)\) functions [RS]
  - **Moderate-degree** \((q < d < n)\) functions
    - \(q=2\): [AKKLR]
    - General \(q\): [KR, JPRZ]
  - Long code/Dictator/Junta testing [BGS, PRS]
  - BCH codes (Trace of low-deg. poly.) [KL]

- **Better Parameters** (motivated by PCPs).
  - \#queries, high-error, amortized query complexity, reduced randomness.
My concerns ...

- Relatively few results ...
  - Why can’t we get “rich” class of properties that are all testable?
  - Why are proofs so specific to property being tested?
- What made Graph Property Testing so well-understood?
- What is “novel” about Property Testing, when compared to “polling”?
Contrast w. Combinatorial P.T.

R is a field F; P is linear!

Universe: \{f: D \rightarrow R\}

Must accept
Don’t care
Must reject

Algebraic Property = Code! (usually)
Basic Implications of Linearity [BHR]

- If $P$ is linear, then:
  - Tester can be made non-adaptive.
  - Tester makes one-sided error
    - $(f \in P \Rightarrow \text{tester always accepts})$.

- Motivates:
  - Constraints:
    - $k$-query test $\Rightarrow$ constraint of size $k$:
      - value of $f$ at $\alpha_1, \ldots, \alpha_k$ constrained to lie in subspace.
  - Characterizations:
    - If non-members of $P$ rejected with positive probability, then $P$ characterized by local constraints.
      - functions satisfying all constraints are members of $P$. 
Pictorially

- $f = \text{assignment to left}$
- Right = constraints
- Characterization of $P$:
  $P = \{f \text{ sat. all constraints}\}$

January 8-10, 2010
ITCS: Invariance in Property Testing
Sufficient conditions?

- Linearity + \( k \)-local characterization  
  \[ \Rightarrow \] \( k \)-local testability?

- [BHR] No!
  - Elegant use of expansion
  - Rule out obvious test; but also any test ... of any \( "q(k)" \)-locality

- Why is characterization insufficient?
  - Lack of symmetry?
Example motivating symmetry

- Conjecture (AKKLR ‘96):
  - Suppose property $P$ is a vector space over $F_2$;
  - Suppose its “invariant group” is “2-transitive”.
  - Suppose $P$ satisfies a $k$-ary constraint
    - $\forall f \in P, f(\alpha_1) + \ldots + f(\alpha_k) = 0$.

- Then $P$ is $(q(k), \varepsilon(k,\delta),\delta)$-locally testable.

- Inspired by “low-degree” test over $F_2$. Implied all previous algebraic tests (at least in weak forms).
Invariances

- Property \( P \) invariant under permutation (function) \( \pi: D \to D \), if
  \[ f \in P \implies f \circ \pi \in P \]

- Property \( P \) invariant under group \( G \) if
  \[ \forall \pi \in G, P \text{ is invariant under } \pi. \]

- Can ask: Does invariance of \( P \) w.r.t. "nice" \( G \) leads to local testability?
Invariances are the key?

- "Polling" works well when (because) invariant group of property is the full symmetric group.

- Modern property tests work with much smaller group of invariances.

- Graph property \sim Invariant under vertex renaming.

- Algebraic Properties & Invariances?
Abstracting Algebraic Properties

- [Kaufman & S.]

- Range is a field $F$ and $P$ is $F$-linear.
- Domain is a vector space over $F$ (or some field $K$ extending $F$).

- Property is invariant under affine (sometimes only linear) transformations of domain.

- “Property characterized by single constraint, and its orbit under affine (or linear) transformations.”
Invariance, Orbits and Testability

- Single constraint implies many
  - One for every permutation $\pi \in \text{Aut}(P)$:
    - “Orbit of a constraint $C$”
      $$ = \{C \circ \pi \mid \pi \in \text{Aut}(P)\} $$

- Extreme case:
  - Property characterized by single constraint +
    its orbit: “Single orbit feature”
    - Most algebraic properties have this feature.
    - W.l.o.g. if domain = vector space over small field.
Example: Degree $d$ polynomials

- **Constraint:** When restricted to a small dimensional affine subspace, function is polynomial of degree $d$ (or less).
  - $\#\text{dimensions} \leq \frac{d}{K - 1}$

- **Characterization:** If a function satisfies above for every small dim. subspace, then it is a degree $d$ polynomial.

- **Single orbit:** Take constraint on any one subspace of dimension $\frac{d}{(K-1)}$; and rotate over all affine transformations.
Some results

- If $P$ is affine-invariant and has $k$-single orbit feature (characterized by orbit of single $k$-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
- Unifies previous algebraic tests (in weak form) with single proof.
Analysis of Invariance-based test

- Property $P$ given by $\alpha_1, \ldots, \alpha_k; V \in F^k$

- $P = \{f | f(A(\alpha_1)) \ldots f(A(\alpha_k)) \in V, \forall \text{ affine } A: K^n \rightarrow K^n\}$

- $\text{Rej}(f) = \text{Prob}_A [ f(A(\alpha_1)) \ldots f(A(\alpha_k)) \text{ not in } V ]$

- Wish to show: If $\text{Rej}(f) < 1/k^3$, then $\delta(f, P) = O(\text{Rej}(f))$. 
BLR Analog

- $\text{Rej}(f) = \Pr_{x,y} [ f(x) + f(y) \neq f(x+y)] < \epsilon$

- Define $g(x) = \text{majority}_y \{\text{Vote}_x(y)\}$, where $\text{Vote}_x(y) = f(x+y) - f(y)$.

- Step 0: Show $\delta(f,g)$ small

- Step 1: $\forall x, \Pr_{y,z} [\text{Vote}_x(y) \neq \text{Vote}_x(z)]$ small.

- Step 2: Use above to show $g$ is well-defined and a homomorphism.
BLR Analysis of Step 1

- Why is $f(x+y) - f(y) = f(x+z) - f(z)$, usually?

<table>
<thead>
<tr>
<th></th>
<th>$f(z)$</th>
<th>$-f(x+z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(y)$</td>
<td>0</td>
<td>$-f(y)$</td>
</tr>
<tr>
<td>$-f(x+y)$</td>
<td>$-f(z)$</td>
<td>$f(x+y+z)$</td>
</tr>
</tbody>
</table>
Generalization

- \( g(x) = \beta \) that maximizes, over \( A \) s.t. \( A(\alpha_1) = x \),
  \( \Pr_A [\beta, f(A(\alpha_2), \ldots, f(A(\alpha_k))] \in V] \)

- Step 0: \( \delta(f, g) \) small.

- \( \text{Vote}_x(A) = \beta \) s.t. \( \beta, f(A(\alpha_2)), \ldots, f(A(\alpha_k)) \in V \)
  (if such \( \beta \) exists)

- Step 1 (key): \( \forall x \), whp \( \text{Vote}_x(A) = \text{Vote}_x(B) \).
- Step 2: Use above to show \( g \in P \).
Matrix Magic?

Say $A(\alpha_1) \ldots A(\alpha_t)$ independent; rest dependent

Random

Doesn’t Matter!
Some results

- If $P$ is affine-invariant and has $k$-single orbit feature (characterized by orbit of single $k$-local constraint); then it is $(k, \delta/k^3, \delta)$-locally testable.
  - Unifies previous algebraic tests with single proof.

- If $P$ is affine-invariant over $K$ and has a single $k$-local constraint, then it is has a $q$-single orbit feature (for some $q = q(K,k)$)
  - (explains the AKKLR optimism)
Results (contd.)

- If $P$ is affine-invariant over $K$ and has a single $k$-local constraint, then it is has a $q$-single orbit feature (for some $q = q(K,k)$)

- Proof Ingredients:
  - Analysis of all affine invariant properties.
  - Rough characterization of locality of constraints, in terms of degrees of polynomials in the family.

- Infinitely many (new) properties ...
More details

- Understanding invariant properties:
  - Recall: all functions from $K^n$ to $F$ are Traces of polynomials
    
    \[
    \text{Trace}(x) = x + x^p + x^{p^2} + \ldots + x^{q/p}
    \]
    
    where $K = F_q$ and $F = F_p$.
  - If $P$ contains $\text{Tr}(3x^5 + 4x^2 + 2)$; then $P$ contains $\text{Tr}(4x^2)$ ...
  - So affine invariant properties characterized by degree of monomials in family.
  - Most of the study ... relate degrees to upper and lower bounds on locality of constraints.
Some results

- If $P$ is affine-invariant over $K$ and has a single $k$-local constraint, then it is has a $q$-single orbit feature (for some $q = q(K,k)$)
  - (explains the AKKLR optimism)
- Unfortunately, $q$ depends inherently on $K$, not just $F$ ... giving counterexample to AKKLR conjecture [joint with Grigorescu & Kaufman]
- Linear invariance when $P$ is not $F$-linear:
  - Abstraction of some aspects of Green’s regularity lemma ... [Bhattacharyyya, Chen, S., Xie]
  - Nice results due to [Shapira]
More results

- Invariance of some standard codes
  - E.g. “dual-BCH”: Have k-single orbit feature! So are “more uniformly” testable.
    
    [Grigorescu, Kaufman, S.]

- Side effect: New (essentially tight) relationships between $\text{Rej}_{\text{AKKL}}(f)$ and $\delta(f,\text{Degree}-d)$ over $F_2$
  
  [with Bhattacharyyya, Kopparty, Schoenebeck, Zuckerman]
More results (contd.)

- Invariance of some standard codes
- Side effect: New (essentially tight) relationships between $\text{Rej}_{\text{AKKL}}(f)$ and $\delta(f,\text{Degree}-d)$ over $\mathbb{F}_2$

- One hope: Could lead to “simple, good locally testable code”?
  - (Sadly, not with affine-inv. [Ben-Sasson, S.])

- Still ... other groups could be used? [Kaufman+Wigderson]
Conclusions

- Invariance seems to be a nice perspective on “property testing” …
  - Certainly helps unify many algebraic property tests.
  - But should be a general lens in sublinear time algorithmics.
Thanks