Invariance in Property Testing

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Based on: works with/of Eli Ben-Sasson, Elena Grigorescu, Tali Kaufman, Shachar Lovett, Ghid Maatouk, Amir Shpilka.
Property Testing

- Sublinear time algorithms:
  - Algorithms running in time $o(\text{input}), o(\text{output})$.
    - Probabilistic.
    - Correct on (approximation) to input.
    - Input given by oracle, output implicit.
  - Crucial to modern context
    - (Massive data, no time).

- Property testing:
  - Restriction of sublinear time algorithms to decision problems (output = YES/NO).
  - Amazing fact: Many non-trivial algorithms exist!
Example 1: Polling

- Is the majority of the population Red/Blue
  - Can find out by random sampling.
  - Sample size $\propto$ margin of error
    - Independent of size of population

- Other similar examples: (can estimate other moments ...)

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Example 2: Linearity

- Can test for homomorphisms:
  - Given: $f: G \rightarrow H$ ($G, H$ finite groups), is $f$ essentially a homomorphism?
  - Test:
    - Pick $x, y$ in $G$ uniformly, ind. at random;
    - Verify $f(x) \cdot f(y) = f(x \cdot y)$

- Completeness: accepts homomorphisms w.p. 1
  - (Obvious)

- Soundness: Rejects $f$ w.p prob. Proportional to its “distance” (margin) from homomorphisms.
  - (Not obvious, [BlumLubyRubinfeld’90])
History (slightly abbreviated)

- [Blum,Luby,Rubinfeld – S’90]
  - Linearity + application to program testing
- [Babai,Fortnow,Lund – F’90]
  - Multilinearity + application to PCPs (MIP).
- [Rubinfeld+S.]
  - Low-degree testing
- [Goldreich,Goldwasser,Ron]
  - Graph property testing
- Since then ... many developments
  - More graph properties, statistical properties, matrix properties, properties of Boolean functions ...
  - More algebraic properties
Pictorial Summary

All properties

Statistical Properties

Boolean functions

Testable!

Not-testable

Linearity

Low-degree

Graph Properties

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Some (introspective) questions

- What is qualitatively novel about linearity testing relative to classical statistics?

- Why are the mathematical underpinnings of different themes so different?

- Why is there no analog of “graph property testing” (broad class of properties, totally classified wrt testability) in algebraic world?
Invariance?

- Property \( P \subseteq \{ f : D \to R \} \)
- Property \( P \) invariant under permutation (function) \( \pi : D \to D \), if
  \[
  f \in P \Rightarrow f \circ \pi \in P
  \]
- Property \( P \) invariant under group \( G \) if
  \[
  \forall \pi \in G, P \text{ is invariant under } \pi.
  \]
- Observation: Different property tests unified/separated by invariance class.
Invariances (contd.)

- Some examples:
  - Classical statistics: Invariant under all permutations.
  - Graph properties: Invariant under vertex renaming.
  - Boolean properties: Invariant under variable renaming.
  - Matrix properties: Invariant under mult. by invertible matrix.
  - Algebraic Properties = ?

- Goals:
  - Possibly generalize specific results.
  - Get characterizations within each class?
  - In algebraic case, get new (useful) codes?
Abstracting Linearity/Low-degree tests

- **Affine Invariance:**
  - Domain = Big field (GF(2^n))
  - or vector space over small field (GF(2)^n).
  - Property invariant under affine transformations of domain (x ↦ A.x + b)

- **Linearity:**
  - Range = small field (GF(2))
  - Property = vector space over range.
Testing Linear Properties

Universe: \( \{f : D \rightarrow R\} \)

R is a field F; P is linear!

Algebraic Property = Code! (usually)

P

Must accept

Don’t care

Must reject
Why study affine-invariance?

- Common abstraction of properties studied in [BLR], [RS], [ALMSS], [AKKLR], [KR], [KL], [JPRZ].
  - (Variations on low-degree polynomials)

- Hopes
  - Unify existing proofs
  - Classify/characterize testability
  - Find new testable codes (w. novel parameters)

- Rest of the talk: Brief summary of findings
Basic terminology

- **Local Constraint:**
  - Example: $f(1) + f(2) = f(3)$.
  - Necessary for testing Linear Properties [BHR]

- **Local Characterization:**
  - Example: $\forall x, y, f(x) + f(y) = f(x+y) \iff f \in P$
  - Aka: LDPC code, k-CNF property etc.
  - Necessary for affine-invariant linear properties.

- **Single-orbit characterization:**
  - One linear constraint + implications by affine-invariance.
  - Feature in all previous algebraic properties.
Affine-invariance & testability

- T-local constraint
- T-characterized
- T-locally testable
- T-S-O-C
State of the art in 2007

- \([\text{AKKLR}]: k\text{-constraint} = k'\text{-testable, for all linear affine-invariant properties?}\)
Affine-invariance & testability

- t-local constraint
- t-characterized
- t-locally testable
- t-S-O-C
Some results

- [Kaufman+S.’07]: Single-orbit $\Rightarrow$ Testable.
Affine-invariance & testability

- t-local constraint
- t-characterized
- t-locally testable
- t-S-O-C [KS’08]
Some results

- [Kaufman+S.’07]: Single-orbit $\Rightarrow$ Testable.
  - Unifies known algebraic testing results.
  - Converts testability to purely algebraic terms.
  - Yields “Constraints = Char. = Testability” for vector spaces over small fields.
  - Left open: Domain = Big field.
  - Exist Many “non-polynomial” testable properties

- [GKS’08]: Over big fields, Constraint $\neq$ Char.
- [BMSS’11]: Over big fields, Char $\neq$ Testability.
- [BGMSS’11]: Many questions/conjectures outlining a possible characterization of affine-invariant properties.
Affine-invariance & testability

- weight-k degrees
- k-local constraint
- k-characterized
- k-locally testable
- k-S-O-C [KS’08]

References:
- [BS’10]
- [BMSS’11]
- [GKS’08]

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Hopes

- Get a complete characterization of locally testable affine-invariant properties.

- Use codes of (polynomially large?) locality to build better LTCs/PCPs?
  - In particular move from “domain = vector space” to “domain = field”.

- More broadly: Apply lens of invariance more broadly to property testing.
Thank You!