Note Title 2/2/2012

INVARIANCE IN PROPERTY TESTING

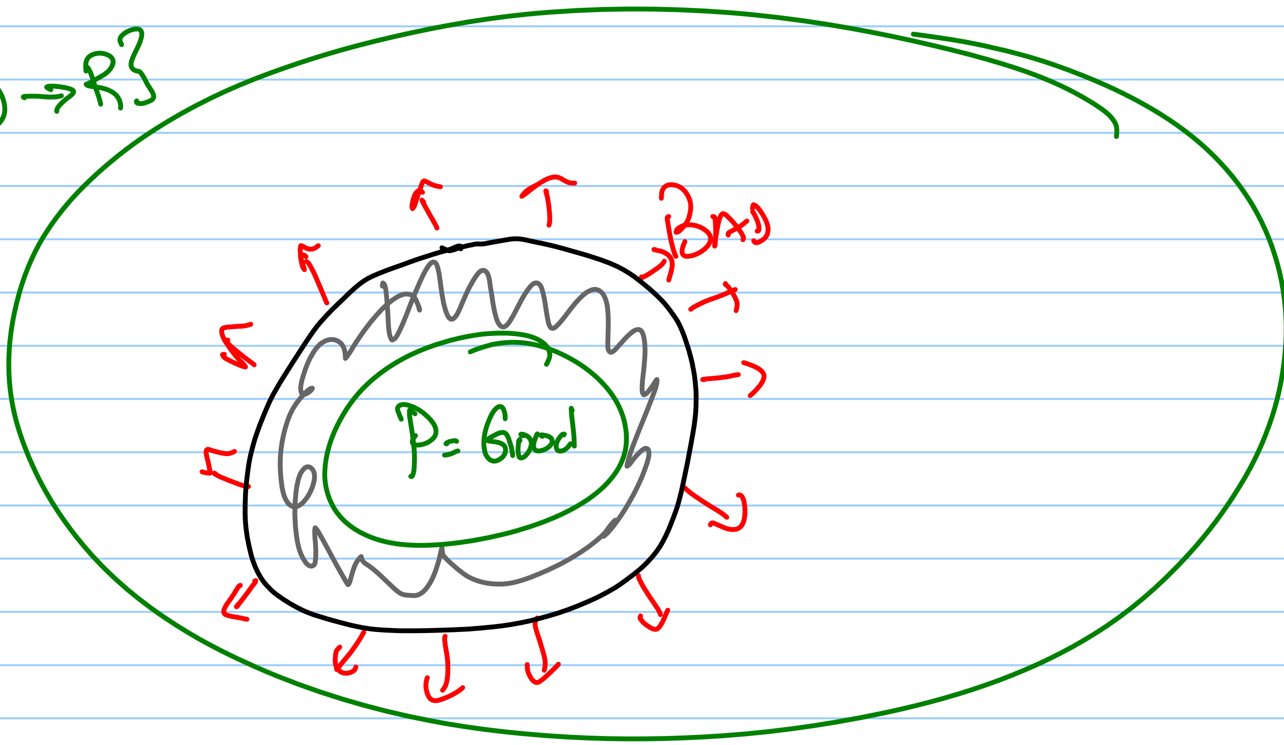
MADHU SUDAN
(MSR)

Agenda

1. Property Testing
2. Invariance Spiel
3. Affine-Invariance
   - Main Results
   - Open Questions

Property testing

$\{D \to R\}$



Given $f: D \to R$

is $f \in P$?

or ϵ-far from P?

Goal: Answer with few queries to f.

INVARIANCE ?

$\pi : D \to D$ permutation.

P invariant under π if $\forall f \in P \quad f \circ \pi \in P$

Invariance class $(P) = \{ \pi \mid P$ invariant under $\pi \}$

Why Invariance?

Constraint: Sequence of domain elements $\alpha_1 \ldots \alpha_k \in D$
Set of legitimate value $S \subseteq R^k$

Tester: Roughly: a collection of constraints ...
But we need many (covering all of D)

$$\boxed{\begin{array}{ccc} \text{Invariance} + 1 \text{ Constraint} & \Longrightarrow & \text{Many Constraints} \\ \alpha_1 \ldots \alpha_k & \Longrightarrow & \pi(\alpha_1), \ldots \pi(\alpha_k) \\ S & & S \end{array}}$$

Invariance of familiar subareas

- Graph Property Testing: Invariant under vertex renaming
- Statistical Properties: Invariant under all permutations
- Boolean properties: Domain $= \{0,1\}^n$
  Invariant under S_n
- Algebraic properties: Domain $= \mathbb{F}_q^n$;
  Invariant under affine transformations

Affine Invariance

[Kaufman + S.]

$\mathbb{F}_q^n \to \mathbb{F}_q$ + linear

$\mathbb{F}_{q^n} \to \mathbb{F}_q$ + linear

Our focus

Bhattacharyya, Grigorescu, Shapira ...

$\mathbb{F}_q^n \to \Sigma$, non-linear

Why $\mathbb{F}_q^n$?

[Kaufman + S '08]: Single-orbit, affine-invariant, linear properties are testable by natural test.

Single-orbit: P characterized by single constraint

+ affine-invariance

Example Property (thanks to [Sergey Yekhanin])

- $P \subseteq \{ \mathbb{F}_{2^n}^2 \to \mathbb{F}_2 \}$
- $P = \{ f \mid \forall \text{ lines } \ell \ \sum_{\ell} f(\ ) = 0 \}$
- dimension $= N - N^{3/4}$; length $= N = (2^n)^2$

  $\sqrt{N}$- locally testable

General hope

- Can find better locally testable codes using affine-invariance
- Step 1: Understand $O(1)$-query testable affine invariant properties.
- Step 2: Understand $\omega(1)$-query testable " .

PROGRESS REPORT - 1

$\mathbb{F}_q^n \to \mathbb{F}_q$ [K.S. '08]: P is $O_q(1)$- query testable

$\Updownarrow$

P has 1- local constraint

$\Updownarrow$

P has no $\omega(1)$- degree polynomial

PROGRESS REPORT - 2

$\mathbb{F}_{q^n} \to \mathbb{F}_q$ - basic

- Low-degree polynomials from $\mathbb{F}_q^n \to \mathbb{F}_q$ are locally testable

  $\to$ can be viewed as function mapping $\mathbb{F}_{q^n} \to \mathbb{F}_q$

  Lemma: [BGMSS] They are single-orbit families

- "Sparse" families are testable [GKS, KL, BSZ]

  $|P| = q^{O(n)}$

Progress Report - 3

$\mathbb{F}_{q^n} \to \mathbb{F}_q$ - composite

- Intersections of single-orbit $P_1 \cap P_2$
- Sums of single-orbit $P_1 + P_2$
- "Lifts" of single-orbit $P : \mathbb{F}_{q^n} \to \mathbb{F}_q$

$$\Big\Downarrow$$

$$\mathrm{Lift}(P) : \mathbb{F}_{q^{n \cdot m}} \to \mathbb{F}_q$$

Limits to Local Testability

- Very little known
- Some special properties [GKS, BMSS]
- A general limitation

$P \subseteq \{ \mathbb{F}_{q^n} \to \mathbb{F}_q \}$ is k-locally testable

$\Rightarrow P \subseteq \{$ degree k-polynomials mapping $\mathbb{F}_q^n \to \mathbb{F}_q \}$

Concrete Questions

① Is local testability $\Longleftrightarrow$ single-orbit characterizability

② Are the set of properties implied by "Progress Report 2 + 3" the only testable properties?

③ If $q=2$, $n=$ prime lifts, intersections are uninteresting.
So is $P \subseteq \{\mathbb{F}_2^{\text{prime}} \to \mathbb{F}_2\}$ testable iff
$P =$ low-degree + Sparse

④ Build PCPs of length $n + n^{1-\delta}$ with n^{ϵ} queries.

The current barrier

- $P \subsetneq \{\deg 2 \text{ poly from } \mathbb{F}_2^n \to \mathbb{F}_2\}$, n prime

  P testable $\Longleftrightarrow$ P sparse ?

- $$\begin{array}{cc} \overset{k}{\longleftrightarrow} & \\ \left[\begin{array}{c} \\ \alpha_i^{2^{d_j}} \\ \\ \end{array}\right] & \updownarrow t \end{array}$$

  $\forall k\ \exists t\ \forall$ prime n

  $\forall \alpha_1 \ldots \alpha_k \in \mathbb{F}_{2^n}$ $\forall d_1, d_2, \ldots d_t$ distinct

  $\Longleftarrow$ M has rank k.

Thank You!