Limits of Local Algorithms in Random Graphs

Madhu Sudan
MSR

Joint work with David Gamarnik (MIT)

Main Result

- Background: Almost surely, random d-regular graph on n vertices has independent set of size $\left(1+o(1)\right)\cdot c_d \cdot n$ for $c_d = \frac{2}{d}\log d$.
- Can you find such a large independent set?
  - Greedy finds one of half this size.

- Our Theorem: “Local algorithms” can not. In fact they fall short by a constant factor.

Definition: Local Algorithms

- Informally: Local algorithms
  - Input = Communication network.
  - Wish to use local communication to compute some property of input.
  - In our case – large independent set in graph.
  - Allowed to use randomness, generated locally.

Formally

- (Randomized) Decision Algorithm:
  - $f(u, G, \vec{w}) \in \{0,1\}$: Determines if $u \in I$.
    - $\vec{w}$ is a weighting, say in $[0,1]$, on vertices
- Correctness:
  - $\forall u, v, G, \vec{w}$ s.t. $u \leftrightarrow_G v$,

$$f(u, G, w) = 0 \text{ or } f(v, G, w) = 0.$$

- Locality:
  - f is r-local if $f(u, G, \vec{w}) = f(v, H, \vec{x})$ whenever r-local weighted neighborhood around u in $(G, \vec{w})$ and v in $(H, \vec{x})$ are identical.

Locality $\neq$ Locality

- Locality in distributed algorithms
  - Usually algorithms try to compute some function of input graph, on the graph itself.
  - Algorithm uses data available topologically locally.
  - Leads to our model
- Locality a la Codes/Property Testing
  - Locality simply refers to number of queries to input.
  - More general model.
  - We can't/don't deal with it.

Motivations for our work

1. Paucity of "complexity" results for random graphs. Major exceptions:
   - Rossman: AC^0/Monotone complexity of planted clique.
   - Feige-Krauthgamer/Meka-Wigderson: SDP relaxations.
2. Physicists explanation of complexity
   - Clustering/Shattering explain inability of algorithms.
3. Graph Limit theory
   - Local characteristics of (random) graphs predict global properties (nearly).

Motivations (contd.)

- Specific conjecture [Hatami-Lovasz-Szegedy]:
  As $r \to \infty$, r-local algorithms should find independent sets of cardinality $c_d(1 - o(1))\, n$.
- Refuted by our theorem.

Proof

- Part I:
  - A clustering phenomenon for independent sets in random graphs [Inspired by Coja-Oglan].
- Part II:
  - Locality $\Rightarrow$ Continuity $\Rightarrow$ $\neg$(Clustering).

Both parts simple.

Clustering Phenomena

- Generally:
  - When you look at "near-optimal" solutions, then they are very structured.
  - $\Rightarrow$ topology of solutions highly disconnected (in Hamming space.
- In our context
  - Consider graph on independent sets (of size $\approx c_d n$) with $I \leftrightarrow J$ if $|I \,\Delta\, J| \leq \epsilon \cdot n$.
  - Highly disconnected?

Clustering Theorem

- Theorem: $\forall d, \exists 0 < \theta < \tau < c_d$ s.t.:
  - Almost surely over G, $\forall I, J$ of size $\approx c_d n$,

$$\frac{|I \cap J|}{n} \notin (\theta, \tau)$$

- Proof:
  - Compute expected number of independent sets with forbidden intersection and note it is $\ll 1$.
  - Second moment proves concentration.
- Implies Clustering.

Locality $\Rightarrow \neg$(Clustering)

- Main Idea:
  - Fix r-local function f, that usually produces independent sets of size $\approx c_d \cdot n$
  - Sample weights twice: $\vec{w}$, and then $\vec{x}$; p-correlatedly.
  - Let $I = f(G, \vec{w})$ and $J = f(G, \vec{x})$.
  - Prove:
    - whp, $|I|, |J| \approx c_d \cdot n$
    - whp, $|I \cap J| \approx \beta(p) \cdot n$
    - $\exists p$ s.t. $\beta(p) \in (\theta, \tau)$

Size of Ind. Set

- Claim: Size of independent set produced by local algorithms is concentrated.
  - Let $\alpha = \alpha(f) = \mathbb{E}_{\vec{w}}[\mathrm{f}(\mathrm{u}, \mathbb{T}_d, \vec{w})]$

    (where $\mathbb{T}_d$ = infinite tree of degree d)
  - W.p. 1-o(1), size of ind. set produced $\approx \alpha \cdot n$.
- Proof:
  - Most neighborhoods are trees ⇒ Expectation.
  - Most neighborhoods are disjoint ⇒ Chebychev.

p-correlated distributions

- Pick $\vec{w}, \vec{y} \in [0,1]^{\wedge}n$, independently.
- Let $\vec{x}_i = \vec{w}_i$ w.p. p and $\vec{y}_i$ otherwise, independently for each i.
- Let $\beta(p) = \mathbb{E}_{\vec{w},\vec{x}}[f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$
- As in previous argument:
  - $\mathbb{E}[|I \cap J|] \approx \beta(p) \cdot n$
  - $|I \cap J|$ concentrated around expectation.

Continuity of $\beta(p)$

- Fix $\vec{w}, \vec{y}$, and consider
$$\Pr[f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$$
- Above expression is some polynomial in p, of degree at most d^r.
- In particular, it is continuous as function of p.
- $\Rightarrow \beta(p)$=Expectation over $\vec{w}, \vec{y}$ is also continuous.
- Suffices to show $[\beta(0), \beta(1)] \cap (\theta, \tau) \neq \emptyset$.

Continuity (contd.)

- $\beta(p) = \mathbb{E}_{\vec{w},\vec{x}}[f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$
- $\beta(1) = \alpha(f) \approx c_d$
- $\beta(0) = \alpha^2 \approx c_d^2$
- Follows from calculations (also naturally) that
  - $[\beta(0), \beta(1)] \cap (\theta, \tau) \neq \emptyset$
- Conclude:
  - whp, $|I|, |J| \approx c_d \cdot n$
  - whp, $|I \cap J| \approx \beta(p) \cdot n$
  - $\exists p$ s.t. $\beta(p) \in (\theta, \tau)$

Conclusions

- "Clustering" is an obstacle?
- Answer:
  - At least to local algorithms.
  - Local algorithms behave continuously, forcing non-clustering of solutions.
- Open questions:
  - Barrier to local algorithms in general sense?
  - To other complexity classes?

Thank You