Communication Amid Uncertainty

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Context in Communication

- Sender + Receiver share (huuuge) context
  - In human comm: Language, news, Social
  - In computer comm: Protocols, Codes, Distributions
  - Helps compress communication

- Perfectly shared $\Rightarrow$ Can be abstracted away.
- Imperfectly shared $\Rightarrow$ What is the cost?
  - How to study?

Communication Complexity

The model (with shared randomness)

x

$f:(x,y) \mapsto \Sigma$

y

$R = \$\$\$$

Alic

Usually studied for lower bounds.

This talk: CC as +ve model.

Bob

$CC(f) =$ # bits exchanged

by best protocol

$f(x,y)$

w.p. 2/3

Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of x and y.
  - More generally: x, y correlated.
- Knowledge of f – function Bob wants to compute
  - may not be exactly known to Alice!
- Shared randomness
  - Alice + Bob may not have identical copies.

Part 1: Uncertain Compression

Classical (One-Shot) Compression

- Sender and Receiver have distribution $P \sim [N]$
- Sender/Receiver agree on Encoder/Decoder E/D
- Sender gets $X \in [N]$; Sends $E(X)$
- Receiver gets $Y = E(X)$; Decodes $\hat{X} = D(Y)$
- Requirement: $\hat{X} = X$ (always)
- Performance: $\mathbb{E}_{X \leftarrow P}[|E(X)|]$

- Trivial Solution: $\mathbb{E}_{X \leftarrow P}[|E(X)|] = \log N$
- Huffman Coding: Achieves $\mathbb{E}_{X \leftarrow P}[|E(X)|] \leq H(P) + 1$

The (Uncertain Compression) problem

[Juba,Kalai,Khanna,S.'11]

- Design encoding/decoding schemes (E/D) s.t.:
  - Sender has distribution $P \sim [N]$
  - Receiver has distribution $Q \sim [N]$
  - Sender gets $X \in [N]$; Sends $E(P,X)$ to receiver.
  - Receiver gets $Y = E(P,X)$; Decodes $\hat{X} = D(Q,Y)$
  - Want: $X = \hat{X}$ (provided P, Q close),

$$\Delta(P,Q) \leq \Delta \text{ if } 2^{-\Delta} \leq \frac{\log P(x)}{\log Q(x)} \leq 2^{\Delta} \text{ for all } x$$

- Motivation: Models natural communication?

Solution (variant of Arith. Coding)

- Uses shared randomness: Sender+Receiver $\leftarrow r \in \{0,1\}^*$
- Use r to define sequences "dictionary"
  - $r_1[1], r_1[2], r_1[3], \ldots$
  - $r_2[1], r$
  - …
  - $r_N[1],$
- Sender … of x
- Receiver … $[1 \ldots L]\}$
- Want: $L : r_z[1 \ldots L] = r_x[1 \ldots L] \Rightarrow Q(z) < Q(x);$

$$\Leftrightarrow (Q(z) > Q(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$$

$$\Leftarrow (P(z) > 4^{-\Delta} P(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$$

Analysis:

$$\mathbb{E}_r[L] = 2\Delta + \log \frac{1}{P(x)}$$

$$\mathbb{E}_{x,r}[L] = 2\Delta + H(P)$$

Implications

- Coding scheme reflects the nature of human communication (extend messages till they feel unambiguous).
- Reflects tension between ambiguity resolution and compression.
  - Larger the ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!
- The "shared randomness'' assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
    - This is problematic.

Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections $\pi, \sigma : [N] \to [N]$
  - $\pi(i)$ = rank of i^{th} player in Alice's ranking.
- Further suppose they know rankings are close.
  - $\forall\, i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $O(\log^* n)$

Part 2: Imperfectly Shared Randomness

Model: Imperfectly Shared Randomness

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where $(r, s) =$ i.i.d. sequence of correlated pairs $(r_i, s_i)_i$; $r_i, s_i \in \{-1, +1\}; \mathbb{E}[r_i] = \mathbb{E}[s_i] = 0; \mathbb{E}[r_i s_i] = \rho \geq 0$.
- Notation:
  - $isr_\rho(f)$ = cc of f with ρ-correlated bits.
  - $cc(f)$: Perfectly Shared Randomness cc. $= isr_1(f)$
  - $priv(f)$: cc with PRIVate randomness $= isr_0(f)$
- Starting point: for Boolean functions f
  - $cc(f) \leq isr_\rho(f) \leq priv(f) \leq cc(f) + \log n$
  - What if $cc(f) \ll \log n$? E.g. $cc(f) = O(1)$

$\rho \leq \tau \Rightarrow$
$isr_\rho(f) \geq isr_\tau(f)$

Imperfectly Shared Randomness: Results

- Model first studied by [Bavarian,Gavinsky,Ito'14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $isr(\text{Equality}) = O(1)$ (among other things)

- Our Results: [Canonne,Guruswami,Meka,S'15]
  - Generally: $cc(f) \leq k \Rightarrow isr(f) \leq 2^k$
  - Converse: $\exists f$ with $cc(f) \leq k$ & $isr(f) \geq 2^k$

Aside: Easy CC Problems [Ghazi,Kamath,S'15]

- Equality testing:
  - $EQ(x,y) = 1 \Leftrightarrow x = y;$ C
- Hamming distance:
  - $H_k(x,y) = 1 \Leftrightarrow \Delta(x,y) \leq k;$
- Small set intersection:
  - $\cap_k (x,y) = 1 \Leftrightarrow$ $\mathrm{wt}(x), \mathrm{wt}(y$
  - $CC(\cap_k) = O(k)$ [Håstad Wi

Protocol:

Fix ECC E $\{0,1\}^n \to \{0,1\}^N$

$poly(k)$ Protoc

Use common

to hash $[n] \to$

$x = (x_1, \ldots, x_n)$
$y = (y_1, \ldots, y_n)$
$\langle x, y \rangle \triangleq \sum_i x_i y_i$

Unstated philosophical contribution of CC a la Yao:

Communication with a focus (“only need to determine $f(x,y)$”)

can be more effective (shorter than $|x|, H(x), H(y), I(x;y)$…)

Equality Testing (our proof)

- Key idea: Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
    - $x = y \Rightarrow \langle X, Y \rangle = N$
    - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$
- Estimating inner products:
  - Building on sketching protocols …
  - Alice: Picks Gaussians $G_1, \ldots G_t \in \mathbb{R}^N$,
  - Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_\rho(1)$ bits suffice if $G \approx_\rho G'$

Gaussian Protocol

General One-Way Communication

- Idea: All communication ≤ Inner Products
- (For now: Assume one-way-cc$(f) \leq k$)
  - For each random string R
    - Alice's message $= i_R \in [2^k]$
    - Bob's output $= f_R(i_R)$ where $f_R: [2^k] \to \{0,1\}$
    - W.p. $\geq \frac{2}{3}$ over R, $f_R(i_R)$ is the right answer.

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- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of f_R).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle; X = (x_R)_R; Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm\epsilon$ with $O_\rho\left(\frac{1}{\epsilon^2}\right)$ communication.

Two-way communication

- Still decided by inner products.

- Simple lemma:
  - $\exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k}$ convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of $\pi_A \in K_A^k, \pi_B \in K_B^k$ equals $\langle \pi_A, \pi_B \rangle$

- Putting things together:

Theorem: $cc(f) \leq k \Rightarrow isr(f) \leq O_\rho(2^k)$

Part 3: Uncertain Functionality

Model

- Bob wishes to compute $f(x,y)$; Alice knows $g \approx f$;
- Alice, Bob given g, f explicitly. (Input size $\sim 2^n$)
- Modelling Questions:
  - What is $\approx$?
  - Is it reasonable to expect to compute $f(x,y)$?
    - E.g., $f(x,y) = f'(x)$? Can't compute $f(x,y)$ without communicating x
- Answers:
  - Assume $x, y \sim \{0,1\}^n \times \{0,1\}^n$ uniformly.
  - $f \approx_\delta g$ if $\delta(f,g) \leq \delta$.
  - Suffices to compute $h(x,y)$ for $h \approx_\epsilon f$

Results - 1

- Thm [Ghazi,Komargodski,Kothari,S.]: $\exists f, g, \mu$ s.t. $\mathrm{cc}_{\mu,.1}^{1way}(f), \mathrm{cc}_{\mu,.1}^{1way}(g) = 1$ and $\delta_\mu(f,g) = o(1)$; but uncertain communication $= \Omega(\sqrt{n})$;
- Thm [GKKS]: But not if $x \perp y$ (in 1-way setting).
  - (2-way, even 2-round, open!)
- Main Idea:
  - Canonical 1-way protocol for f:
    - Alice + Bob share random $y_1, \ldots y_m \in \{0,1\}^n$.
    - Alice sends $f(x, y_1), \ldots, f(x, y_m)$ to Bob.
    - Protocol used previously … but not as "canonical".
  - Canonical protocol robust when $f \approx g$.

Conclusions

- Positive view of communication complexity: Communication with a focus can be effective!
- Context Important:
  - New layer of uncertainty.
  - New notion of scale (context LARGE)
    - Importance of $o(\log n)$ additive factors.
- Many "uncertain" problems can be solved without resolving the uncertainty (which is a good thing)
- Many open directions+questions

Thank You!