Imperfectly Shared Randomness in Communication

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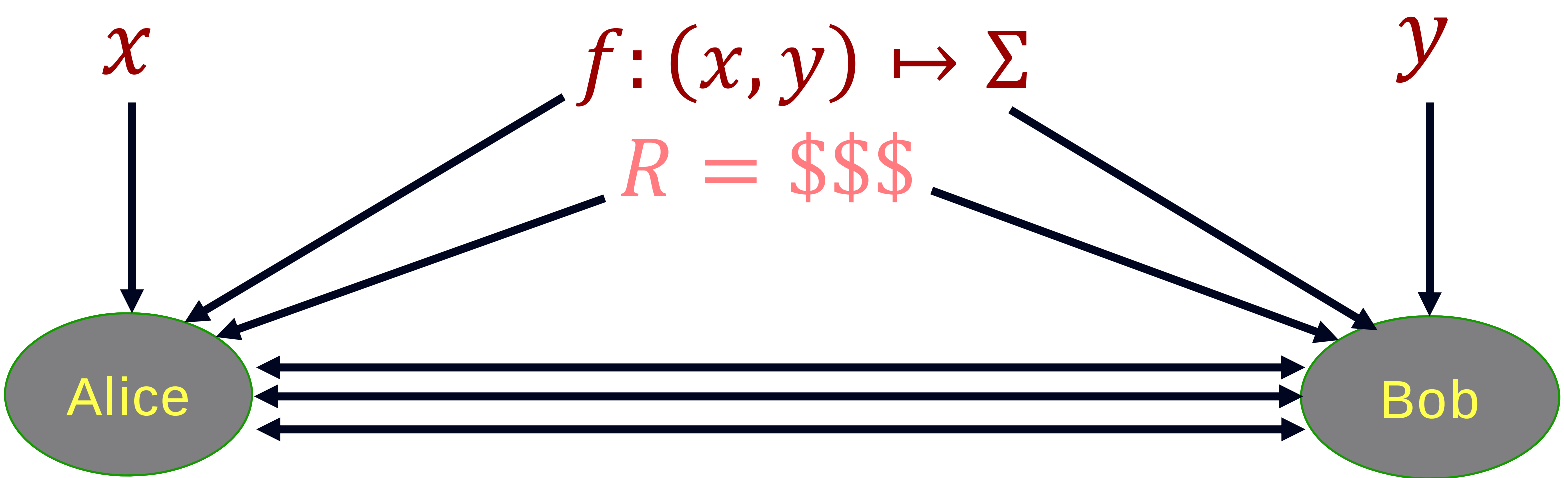
Communication Complexity

The model (with shared randomness)



$CC(f) =$ # bits exchanged
by best protocol

$f(x, y)$
w.p. 2/3

Communication Complexity: Motivation

- Lower bounds:
  - Circuit complexity, Streaming, Data Structures, extended formulations …
- Upper bounds?
  - What is the right model for Communication (e.g., this talk)? - Shannon'48 or Yao'79?
    - If you wish to reproduce this talk …
      - Shannon '48
    - If goal is for you to learn something, or if we expect to use interaction …
      - Yao '79!!

Natural (Contextual) communication

- Communication among humans:
  - Large context.
  - (Small) uncertainty about context.
  - Short communications.
- Can we use CC to study such communication?
  - What are example problems?
  - What are reliability mechanisms?
  - How do you leverage small uncertainty about large context?
- What are examples of problems with small communication complexity?

Aside: Easy CC Problems

- Equality testing:
  - $EQ(x,y) = 1 \Leftrightarrow x = y$; C
- Hamming distance:
  - $H_k(x,y) = 1 \Leftrightarrow \Delta(x,y) \le k$;
- Small set intersection:
  - $\cap_k (x,y) = 1 \Leftrightarrow$ $\mathrm{wt}(x), \mathrm{wt}(y$
  - $CC(\cap_k) = O(k)$ [Håstad Wi
- Gap (Real) Inner Produ
  - $x, y \in \mathbb{R}^n$; $|x|_2, |y|_2 = 1$;
  - $GIP_{c,s}(x,y) = 1$ if $\langle x,y \rangle \ge c$;

Protocol:

Fix ECC E: $\{0,1\}^n \to \{0,1\}^N$

$poly(k)$ Protoc

Use common

to hash $[n] \to$

$X = (X_1, \ldots, X_N)$

$Y = (Y_1, \ldots, Y_N)$

$\langle X, Y \rangle \triangleq \sum_i X_i Y_i$

Main Insight:

If $G \leftarrow N(0,1)^n$, then

$\mathbb{E}[\langle G, x \rangle \cdot \langle G, y \rangle] = \langle x, y \rangle$

Thanks to Badih Ghazi and Pritish Kamath

Uncertainty in Communication

- Overarching question: Are there communication mechanisms that can overcome uncertainty?

- What is uncertainty? Some possible models
  - Bob wishes to compute f. Alice only has "approximate" knowledge of f.
  - Alice & Bob's inputs are strongly correlated.

- This talk: Alice, Bob don't share randomness perfectly; only approximately.

Rest of this talk

- Model: Imperfectly Shared Randomness
- Positive results: Coping with imperfectly shared randomness.
- Negative results: Analyzing weakness of imperfectly shared randomness.

Model: Imperfectly Shared Randomness

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where $(r, s) =$ i.i.d. sequence of correlated pairs $(r_i, s_i)_i$; $r_i, s_i \in \{-1, +1\}; \mathbb{E}[r_i] = \mathbb{E}[s_i] = 0; \mathbb{E}[r_i s_i] = \rho \geq 0$.
- Notation:
  - $isr_\rho(f)$ = cc of f with ρ-correlated bits.
  - $cc(f)$: Perfectly Shared Randomness cc. $= isr_1(f)$
  - $priv(f)$: cc with PRIVate randomness $= isr_0(f)$
- Starting point: for Boolean functions f
  - $cc(f) \leq isr_\rho(f) \leq priv(f) \leq cc(f) + \log n$
  - What if $cc(f) \ll \log n$? E.g. $cc(f) = O(1)$

$\rho \leq \tau \Rightarrow$
$isr_\rho(f) \geq isr_\tau(f)$

Results

- Model first studied by [Bavarian,Gavinsky,Ito'14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $isr(\text{Equality}) = O(1)$ (among other things)

- Our Results:
  - Generally: $cc(f) \leq k \Rightarrow isr(f) \leq 2^k$
  - Converse: $\exists f \text{ with } cc(f) \leq k \ \& \ isr(f) \geq 2^k$

Equality Testing (our proof)

- Key idea: Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
    - $x = y \Rightarrow \langle X, Y \rangle = N$
    - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$
- Estimating inner products:
  - Building on sketching protocols …
  - Alice: Picks Gaussians $G_1, \ldots G_t \in \mathbb{R}^N$,
  - Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_\rho(1)$ bits suffice if $G \approx_\rho G'$

Gaussian Protocol

General One-Way Communication

- Idea: All communication $\leq$ Inner Products
- (For now: Assume one-way-cc$(f) \leq k$)
  - For each random string R
    - Alice's message = $i_R \in [2^k]$
    - Bob's output = $f_R(i_R)$ where $f_R: [2^k] \to \{0,1\}$
    - W.p. $\geq \frac{2}{3}$ over R, $f_R(i_R)$ is the right answer.

General One-Way Communication

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  - W.p. $\geq \frac{2}{3}$, $f_R(i_R)$ is the right answer.
- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of f_R).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle; X = (x_R)_R; Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm\epsilon$ with $O_\rho\left(\frac{1}{\epsilon^2}\right)$ communication.

Two-way communication

- Still decided by inner products.

- Simple lemma:
  - $\exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k}$ convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of $\pi_A \in K_A^k, \pi_B \in K_B^k$ equals $\langle \pi_A, \pi_B \rangle$

- Putting things together:

Theorem: $cc(f) \leq k \Rightarrow isr(f) \leq O_\rho(2^k)$

Main Technical Result: Matching lower bound

Theorem: There exists a (promise) problem f s.t.

- $cc(f) \leq k$
- $isr_\rho(f) \geq \exp(k)$

- The Problem:
  - Gap Sparse Inner Product (G-Sparse-IP).
  - Alice gets sparse $x \in \{0,1\}^n$; $\mathrm{wt}(x) \approx 2^{-k} \cdot n$
  - Bob gets $y \in \{0,1\}^n$
  - Promise: $\langle x, y\rangle \geq (.9)2^{-k} \cdot n$ or $\langle x, y\rangle \leq (.6)2^{-k} \cdot n$
  - Decide which.

G-Sparse-IP:
$\boldsymbol{x, y \in \{0,1\}^n; wt(x) \approx 2^{-k} \cdot n}$
Decide $\boldsymbol{\langle x, y\rangle \geq (.9)\, 2^{-k} \cdot n}$
or $\boldsymbol{\langle x, y\rangle \leq (.6)\, 2^{-k} \cdot n}$**?**

psr Protocol for G-Sparse-IP

- Note: Gaussian protocol takes $O(2^k)$ bits.
  - Need to get exponentially better.
- Idea: $x_i \neq 0 \;\Rightarrow\; y_i$ correlated with answer.
- Use (perfectly) shared randomness to find random index i s.t. $x_i \neq 0$.
- Shared randomness: $i_1, i_2, i_3, \ldots$ uniform over $[n]$
- Alice → Bob: smallest index j s.t. $x_{i_j} \neq 0$.
- Bob: Accept if $y_{i_j} = 1$
- Expect $j \approx 2^k$; $cc \leq k$.

G-Sparse-IP:
$x, y \in \{0,1\}^n; wt(x) \approx 2^{-k} \cdot n$
Decide $\langle x, y \rangle \geq (.9)\, 2^{-k} \cdot n$
or $\langle x, y \rangle \leq (.6)\, 2^{-k} \cdot n$?

Towards a lower bound: Ruling out a natural approach

- Natural approach:
  - Alice and Bob use (many) correlated bits to agree perfectly on few random bits?
  - For G-Sparse-IP need $O(2^k \log n)$ random bits.
- Agreement Distillation Problem:
  - Alice & Bob exchange t bits; generate k random bits, with agreement probability γ.
  - Lower bound [Bogdanov, Mossel]:

$$t \geq k - O\left(\log \frac{1}{\gamma}\right)$$

Towards Lower Bound

- Explaining two natural protocols:
  - Gaussian Inner Product Protocol:
    - Ignore sparsity and just estimate inner product.
    - Uses $\sim 2^{2k}$ bits. Need to prove it can't be improved!

G-Sparse-IP:
$\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n; \boldsymbol{wt}(\boldsymbol{x}) \approx 2^{-k} \cdot \boldsymbol{n}$
Decide $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq (.9)\, 2^{-k} \cdot \boldsymbol{n}$
or $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq (.6)\, 2^{-k} \cdot \boldsymbol{n}$**?**

Optimality of Gaussian Protocol

- Problem:
  - $(x, y) \leftarrow \mu^n$: $\mu = \mu_{YES}$ *or* μ_{NO} supported on $\mathbb{R} \times \mathbb{R}$

    μ_{YES}: ϵ-correlated Gaussians

    μ_{NO}: uncorrelated Gaussians

- Lemma: Separating μ_{YES}^n *vs.* μ_{NO}^n requires $\Omega(\epsilon^{-1})$ bits of communication.

- Proof: Reduction from Disjointness

- Conclusion: Can't ignore sparsity!

G-Sparse-IP:
$x, y \in \{0,1\}^n; wt(x) \approx 2^{-k} \cdot n$
Decide $\langle x, y \rangle \geq (.9)\, 2^{-k} \cdot n$
or $\langle x, y \rangle \leq (.6)\, 2^{-k} \cdot n$**?**

Towards Lower Bound

- Explaining two natural protocols:
  - Gaussian Inner Product Protocol:
    - Ignore sparsity and just estimate inner product.
    - Uses $\sim 2^{2k}$ bits. Need to prove it can't be improved!
  - Protocol with perfectly shared randomness:
    - Alice & Bob agree on coordinates to focus on:

$$(i_1, i_2, \dots, i_{2^k}, \dots);$$

      - Either i_1 has high entropy (over choice of r, s)
        - Violates agreement distillation bound
      - Or has low-entropy:
        - Fix distributions of x, y s.t. $x_{i_1} \perp y_{i_1}$

G-Sparse-IP:
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or $\langle x, y \rangle \leq (.6)\, 2^{-k} \cdot n$**?**

Aside: Distributional lower bounds

- Challenge:
  - Usual CC lower bounds are distributional.
  - cc(G-Sparse-IP) $\leq k$, $\forall$ inputs.

    $\Rightarrow cc$(G-Sparse-IP) $\leq k$ $\forall$ distributions.

    $\Rightarrow$ *det-cc* (G-Sparse-IP) $\leq k$ $\forall$ distributions.
- So usual approach can't work ...
  - Need to fix strategy first and then "identify" a hard distribution for the strategy ...

G-Sparse-IP:
$x, y \in \{0,1\}^n; wt(x) \approx 2^{-k} \cdot n$
Decide $\langle x, y \rangle \geq (.9)\, 2^{-k} \cdot n$
or $\langle x, y \rangle \leq (.6)\, 2^{-k} \cdot n$**?**

Towards lower bound

- Summary so far:
  - Symmetric strategy ⇒ 2^k bits of comm.
  - Strategy asymmetric; $x_1, y_1 \ldots x_k, y_k$ have high influence ⇒ fix the distribution so these coordinates do not influence answer.
  - Strategy asymmetric; with random coordinate having high influence ⇒ violates agreement lower bound.
- Are these exhaustive? How to prove this?
  - Invariance Principle!!

    [Mossel, O'Donnell, Oleskiewisz], [Mossel] ...

ISR lower bound for GSIP.

- One-way setting (for now)
- Strategies: Alice $f_r(x) \in [K]$; Bob $g_s(y) \in \{0,1\}^K$;
- Distributions:
  - If x_i, y_i have high influence on (f_r, g_s) w.h.p. over (r, s) then set $x_i = y_i = 0$. [i is BAD]
  - Else y_i correlated with x_i in YES case, and independent in NO case.
- Analysis:
  - $i \in BAD$ influential in both $f_r, g_s \Rightarrow$ No help.
  - $i \notin BAD$ influential ... $\Rightarrow$ violates agreement lower bound.
  - No common influential variable
    $\Rightarrow x, y$ can be replaced by Gaussians
    $\Rightarrow 2^k$ bits needed.

G-Sparse-IP:
$\boldsymbol{x, y \in \{0,1\}^n; wt(x) \approx 2^{-k} \cdot n}$
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Invariance Principle + Challenges

- Informal Invariance Principle: f, g low-degree polynomials with no common influential variable $\Rightarrow \mathrm{Exp}_{x,y}[f(x)g(y)] \approx \mathrm{Exp}_{X,Y}[f(X)g(Y)]$ (caveat $f \approx f; g \approx g$)
  - where x, y Boolean n-wise product dist.
  - and X, Y Gaussian n-wise product dist
- Challenges [+ Solutions]:
  - Our functions not low-degree [Smoothening]
  - Our functions not real-valued
    - $g: \{0,1\}^n \to \{0,1\}^\ell$: [Truncate range to $[0,1]^\ell$]
    - $f: \{0,1\}^n \to [\ell]$: [???, [work with $\Delta(\ell)$]]

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- Challenges
  - Our functions not low-degree [Smoothening]
  - Our functions not real-valued [Truncate]
  - Quantity of interest is not $f(x) \cdot g(y)$ ...
    - [Can express quantity of interest as inner product.]
  - ... (lots of grunge work ...)
- Get a relevant invariance principle (next)

Invariance Principle for CC

Theorem: For every convex $K_1, K_2 \subseteq [-1,1]^\ell$
$\exists$ transformations T_1, T_2 s.t.
if $f : \{0,1\}^n \to K_1$ and $g : \{0,1\}^n \to K_2$
have no common influential variable, then
$F = T_1 f : \mathbb{R}^n \to K_1$ and $G = T_2 g : \mathbb{R}^n \to K_2$ satisfy
$\mathrm{Exp}_{x,y}[\langle f(x), g(y)\rangle] \approx \mathrm{Exp}_{X,Y}[\langle F(X), G(Y)\rangle]$

- Main differences: f, g vector-valued.
- Functions are transformed: $f \mapsto F; g \mapsto G$
- Range preserved exactly $(K_1 = \Delta(\ell); K_2 = [0,1]^\ell)$!
    - So F, G are still communication strategies!

Summarizing

- k bits of comm. with perfect sharing
  $\rightarrow 2^k$ bits with imperfect sharing.
- This is tight
- Invariance principle for communication
  - Agreement distillation
  - Low-influence strategies

G-Sparse-IP:
$\boldsymbol{x, y} \in \{\mathbf{0, 1}\}^{\boldsymbol{n}}; \boldsymbol{wt(x)} \approx \mathbf{2}^{-\boldsymbol{k}} \cdot \boldsymbol{n}$
Decide $\langle \boldsymbol{x, y} \rangle \geq (.\mathbf{9})\ \mathbf{2}^{-\boldsymbol{k}} \cdot \boldsymbol{n}$
or $\langle \boldsymbol{x, y} \rangle \leq (.\mathbf{6})\ \mathbf{2}^{-\boldsymbol{k}} \cdot \boldsymbol{n}$**?**

Conclusions

- Imperfect agreement of context important.
  - Dealing with new layer of uncertainty.
  - Notion of scale (context LARGE)
- Many open directions+questions:
  - Imperfectly shared randomness:
    - One-sided error?
    - Does interaction ever help?
    - How much randomness?
    - More general forms of correlation?

Thank You!