Communication Amid Uncertainty

Madhu Sudan
Harvard University

Based on joint works with Brendan Juba, Oded Goldreich, Adam Kalai, Sanjeev Khanna, Elad Haramaty, Jacob Leshno, Clement Canonne, Venkatesan Guruswami, Badih Ghazi, Pritish Kamath, Ilan Komargodski and Pravesh Kothari.

Obligatory Sales Pitch

- Most of communication theory [a la Shannon, Hamming]:
  - Built around sender and receiver perfectly synchronized.
- Most Human communication ...
  - ... does not assume perfect synchronization.
- And increasingly ... device-device communication can also not rely on this.
- Can we build a mathematical theory of imperfectly synchronized communication?
  - What are questions/answers ?

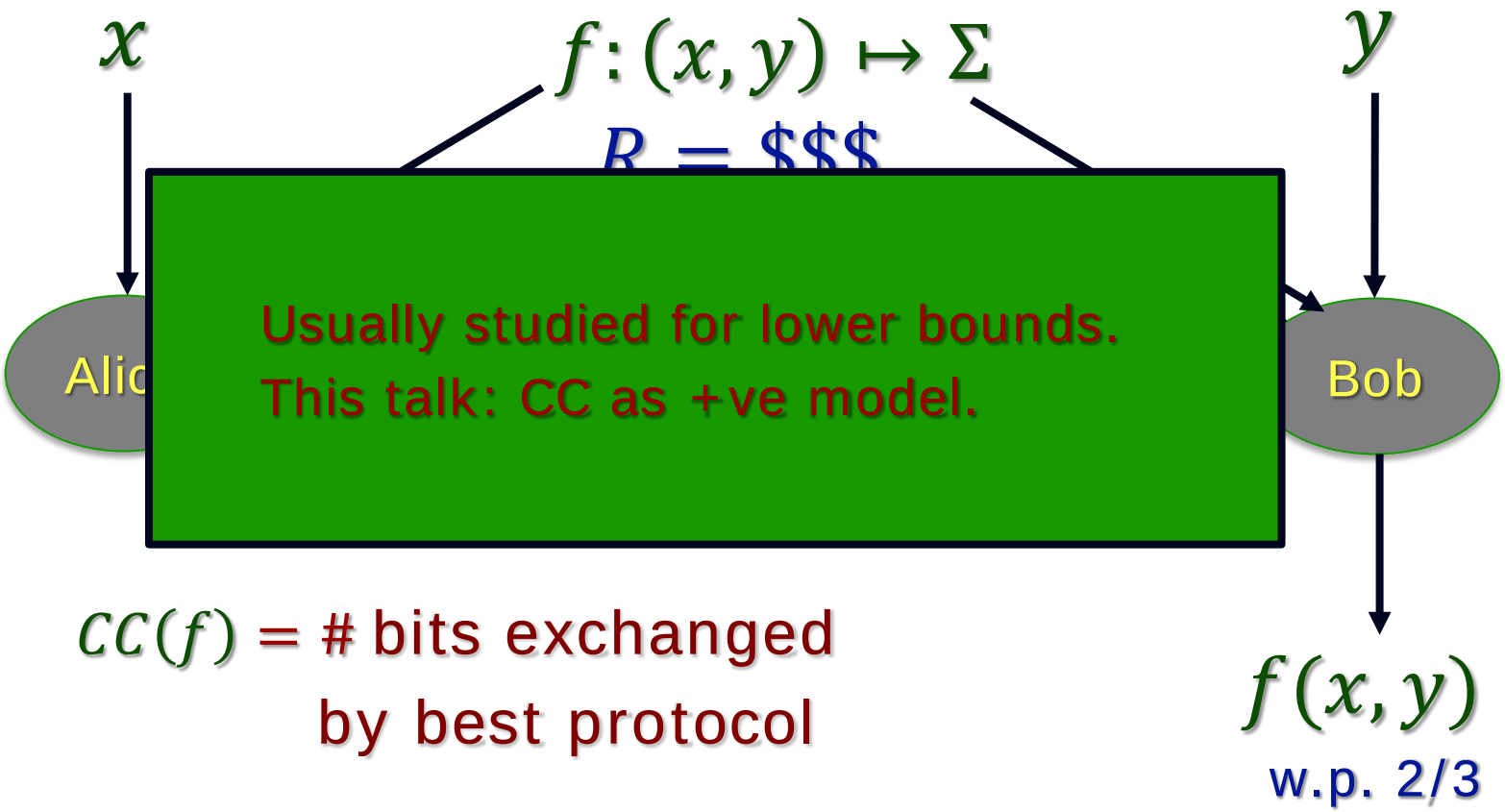
Context in Communication

- In most forms of communication:
  Sender + Receiver share (huuuge) context
  - In human comm: Language, news, Social
  - In computer comm: Protocols, Codes, Distributions
  - Helps compress communication

- Perfectly shared ⇒ Can be abstracted away.
- Imperfectly shared ⇒ What is the cost?
  - How to study?

Communication Complexity

The model (with shared randomness)



$CC(f) = \#$ bits exchanged by best protocol

Aside: Easy CC Problems [Ghazi,Kamath,S'15]

∃ Problems with large inputs and small communication?

- Equality testing:
  - $EQ(x,y) = 1 \Leftrightarrow x = y$; C
- Hamming distance:
  - $H_k(x,y) = 1 \Leftrightarrow \Delta(x,y) \leq k$;
- Small set intersection:
  - $\cap_k(x,y) = 1 \Leftrightarrow$ wt(x), wt$(y$
  - $CC(\cap_k) = O(k)$ [Håstad Wi

Protocol:

Fix ECC $E: \{0,1\}^n \to \{0,1\}^N$

$poly(k)$ Proto

Use common

to hash $[n] \to$

$x = (x_1, \ldots, x_n)$
$y = (y_1, \ldots, y_n)$
$\langle x, y \rangle \triangleq \sum_i x_i y_i$

Unstated philosophical contribution of CC a la Yao:

Communication with a <u>focus</u> ("only need to determine $f(x,y)$")

can be more <u>effective</u> (shorter than $|x|, H(x), H(y), I(x;y) \ldots$)

Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of x and y.
  - More generally: x, y correlated.
- Knowledge of f – function Bob wants to compute
  - may not be exactly known to Alice!
- Shared randomness
  - Alice + Bob may not have identical copies.

1

3

2

Part 1: Uncertain Compression

Classical (One-Shot) Compression

- Sender and Receiver have distribution $P \sim [N]$
- Sender/Receiver agree on Encoder/Decoder E/D
- Sender gets $X \in [N]$; Sends $E(X)$
- Receiver gets $Y = E(X)$; Decodes $\hat{X} = D(Y)$
- Requirement: $\hat{X} = X$ (always)
- Performance: $\mathbb{E}_{X \leftarrow P}[|E(X)|]$

- Trivial Solution: $\mathbb{E}_{X \leftarrow P}[|E(X)|] = \log N$
- Huffman Coding: Achieves $\mathbb{E}_{X \leftarrow P}[|E(X)|] \leq H(P) + 1$

The (Uncertain Compression) problem

[Juba,Kalai,Khanna,S.'11]

- Design encoding/decoding schemes (E/D) s.t.:
  - Sender has distribution $P \sim [N]$
  - Receiver has distribution $Q \sim [N]$
  - Sender gets $X \in [N]$; Sends $E(P,X)$ to receiver.
  - Receiver gets $Y = E(P,X)$; Decodes $\hat{X} = D(Q,Y)$
  - Want: $X = \hat{X}$ (provided P,Q close),

$$\Delta(P,Q) \leq \Delta \text{ if } \left|\log \frac{P(x)}{Q(x)}\right| \leq \Delta \text{ for all } x$$

- Motivation: Models natural communication?

Solution (variant of Arith. Coding)

- Uses shared randomness: Sender+Receiver $\leftarrow r \in \{0,1\}^*$
- Use r to define sequences "dictionary"
  - $r_1[1], r_1[2], r_1[3], \ldots$
  - $r_2[1], r$
  - …
  - $r_N[1],$
- Sender of x
- Receiver $x[1 \ldots L]\}$
- Want: $L: r_z[1 \ldots L] = r_x[1 \ldots L] \Rightarrow Q(z) < Q(x);$

$$\Leftrightarrow (Q(z) > Q(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$$

$$\Leftarrow (P(z) > 4^{-\Delta} P(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$$

Analysis:

$$\mathbb{E}_r[L] = 2\Delta + \log \frac{1}{P(x)}$$

$$\mathbb{E}_{x,r}[L] = 2\Delta + H(P)$$

Implications

- Coding scheme reflects the nature of human communication (extend messages till they feel unambiguous).
- Reflects tension between ambiguity resolution and compression.
  - Larger the ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!
- The “shared randomness” assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
    - This is problematic.

Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N movies.
  - Rankings = bijections $\pi, \sigma : [N] \to [N]$
  - $\pi(i)$ = rank of ith movie in Alice's ranking.
- Further suppose they know rankings are close.
  - $\forall\, i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $O(\log^* n)$

Open Questions (Compression)

- Best Deterministic Uncertain Compression?
  - Best known: $O(H(P) + \log\log N)$
  - Dependence on N?
  - Leading constant?
- Does Private Randomness help?
  - Can we do $O(H(P) + \log\log\log N)$?
- Movie ranking problem:
  - Dependence on N necessary?

    $\Rightarrow$ Compression length w. Det/Priv. Randomness grows with N

Part 2: Imperfectly Shared Randomness

Model: Imperfectly Shared Randomness

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where
  (r, s) = i.i.d. sequence of correlated pairs $(r_i, s_i)_i$;
  $r_i, s_i \in \{-1, +1\}; \mathbb{E}[r_i] = \mathbb{E}[s_i] = 0; \mathbb{E}[r_i s_i] = \rho \geq 0$.
- Notation:
  - $isr_\rho(f)$ = cc of f with ρ-correlated bits.
  - $cc(f)$: Perfectly Shared Randomness cc. $= isr_1(f)$
  - $priv(f)$: cc with PRIVate randomness $= isr_0(f)$
- Starting point: for Boolean functions f
  - $cc(f) \leq isr_\rho(f) \leq priv(f) \leq cc(f) + \log n$
  - What if $cc(f) \ll \log n$? E.g. $cc(f) = O(1)$

$\rho \leq \tau \Rightarrow$
$isr_\rho(f) \geq isr_\tau(f)$

Imperfectly Shared Randomness: Results

- Model first studied by [Bavarian,Gavinsky,Ito'14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $isr(\text{Equality}) = O(1)$ (among other things)

- Our Results: [Canonne,Guruswami,Meka,S'15]
  - Generally: $cc(f) \leq k \Rightarrow isr(f) \leq 2^k$
  - Converse: $\exists f$ with $cc(f) \leq k$ & $isr(f) \geq 2^k$

Equality Testing (our proof)

- Key idea: Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
    - $x = y \Rightarrow \langle X, Y \rangle = N$
    - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$
- Estimating inner products:
  - Building on sketching protocols …
  - Alice: Picks Gaussians $G_1, \ldots G_t \in \mathbb{R}^N$,
  - Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_\rho(1)$ bits suffice if $G \approx_\rho G'$

Gaussian Protocol

General One-Way Communication

- Idea: All communication $\leq$ Inner Products
- (For now: Assume one-way-cc$(f) \leq k$)
  - For each random string R
    - Alice's message $= i_R \in [2^k]$
    - Bob's output $= f_R(i_R)$ where $f_R: [2^k] \to \{0,1\}$
    - W.p. $\geq \frac{2}{3}$ over R, $f_R(i_R)$ is the right answer.

General One-Way Communication

- For each random string R
  - Alice's message $= i_R \in [2^k]$
  - Bob's output $= f_R(i_R)$ where $f_R: [2^k] \to \{0,1\}$
  - W.p. $\geq \frac{2}{3}$, $f_R(i_R)$ is the right answer.
- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of f_R).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle; X = (x_R)_R; Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm\epsilon$ with $O_\rho\left(\frac{1}{\epsilon^2}\right)$ communication.

Two-way communication

- Still decided by inner products.

- Simple lemma:
  - $\exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k}$ convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of $\pi_A \in K_A^k, \pi_B \in K_B^k$ equals $\langle \pi_A, \pi_B \rangle$

- Putting things together:

Theorem: $cc(f) \leq k \Rightarrow isr(f) \leq O_\rho(2^k)$

The Tightness Example

- Sparse Gap Inner Product:
  - Alice $\leftarrow x \in \{0,1\}^n$; $\mathrm{wt}(x) \leq 2^{-k} \cdot n$ (Sparse)
  - Bob $\leftarrow y \in \{-1,1\}^n$;
  - Decide: $\langle x, y \rangle \geq (.9)2^{-k}n$ or $\langle x, y \rangle \leq 0$

- Shared randomness protocol:
  - Alice communicates random bit i s.t. $x_i = 1$
  - Bob outputs y_i

- Lower bound:
  - Invariance principle, Gap Hamming Distance

Open Questions (I.S.R.)

- Exponential gap for total function?
  - $\exists f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$, with
    $$cc(f) \leq k, \qquad \& \qquad isr - cc(f) \geq 2^k$$
- Level of correlation?
  - $\exists f: \{0,1\}^n \times \{0,1\}^n \to \{0,1,?\}$, with
    $$isr_{.9}(f) \leq k, \qquad \& \qquad isr_{.1}(f) \geq 2^k$$
- Does interaction help if randomness is not perfectly shared?

Part 3: Uncertain Functionality

Model

- Bob wishes to compute $f(x,y)$; Alice knows $g \approx f$;
- Alice, Bob given g, f explicitly. (Input size $\sim 2^n$)
- Modelling Questions:
  - What is $\approx$?
  - Is it reasonable to expect to compute $f(x,y)$?
    - E.g., $f(x,y) = f'(x)$? Can't compute $f(x,y)$ without communicating x
- Answers:
  - Assume $x, y \sim \{0,1\}^n \times \{0,1\}^n$ uniformly.
  - $f \approx_\delta g$ if $\delta(f,g) \leq \delta$.
  - Suffices to compute $h(x,y)$ for $h \approx_\epsilon f$

Results - 1

- Thm [Ghazi,Komargodski,Kothari,S.]: $\exists f, g, \mu$ s.t. $\mathrm{cc}^{1way}_{\mu,.1}(f), \mathrm{cc}^{1way}_{\mu,.1}(g) = 1$ and $\delta_\mu(f, g) = o(1)$; but uncertain communication $= \Omega(\sqrt{n})$;
- Thm [GKKS]: But not if $x \perp y$ (in 1-way setting).
  - (2-way, even 2-round, open!)
- Main Idea:
  - Canonical 1-way protocol for f:
    - Alice + Bob share random $y_1, \ldots y_m \in \{0,1\}^n$.
    - Alice sends $f(x, y_1), \ldots, f(x, y_m)$ to Bob.
    - Protocol used previously … but not as "canonical".
  - Canonical protocol robust when $f \approx g$.

Open Questions (Uncertain Functionality)

- What happens when x, y correlated?
  - [Ghazi-S.] ∃functions where cc grows with $I(x;y)$
    - Exact dependence?
    - Is $I(x;y)$ right measure?
- What happens to communication with multiple rounds?
  - Two rounds?
- What is the right task that captures uncertainty in natural communication?

Conclusions

- Positive view of communication complexity: Communication with a focus can be effective!
- Context Important:
  - New layer of uncertainty.
  - New notion of scale (context LARGE)
    - Importance of $o(\log n)$ additive factors.
- Many "uncertain" problems can be solved without resolving the uncertainty (which is a good thing)
- Many open directions+questions

Thank You!