Communication Amid Uncertainty

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Obligatory Sales Pitch

- Most of communication theory [a la Shannon, Hamming]:
  - Built around sender and receiver perfectly synchronized.
- Most Human communication ...
  - ... does not assume perfect synchronization.
- And increasingly ... device-device communication can also not rely on this.
- Can we build a mathematical theory of imperfectly synchronized communication?
  - What are questions/answers?
Context in Communication

- In most forms of communication: Sender + Receiver share (huuuge) context
  - In human comm: Language, news, Social
  - In computer comm: Protocols, Codes, Distributions
  - Helps compress communication

- Perfectly shared $\Rightarrow$ Can be abstracted away.
- Imperfectly shared $\Rightarrow$ What is the cost?
  - How to study?
Communication Complexity

The model (with shared randomness)

\[ f: (x, y) \mapsto \Sigma \]

\[ R = $$$ \]

Usually studied for lower bounds.
This talk: CC as +ve model.

\[ CC(f) = \# \text{ bits exchanged} \]
by best protocol

\[ f(x, y) \text{ w.p. } 2/3 \]
Aside: Easy CC Problems [Ghazi,Kamath,S’15]

∃ Problems with large inputs and small communication?

- Equality testing:
  - $EQ(x, y) = 1 \iff x = y; \quad CC(EQ) = O(1)$

- Hamming distance:
  - $H_k(x, y) = 1 \iff \Delta(x, y) \leq k;
  \quad CC(H_k) = O(k \log k)$ [Huang et al.]

- Small set intersection:
  - $\cap_k (x, y) = 1 \iff \text{wt}(x), \text{wt}(y) \leq k$
  - $CC(\cap_k) = O(k)$ [Håstad, Wigderson]

Protocol:

1. Fix ECC $E: \{0,1\}^n \rightarrow \{0,1\}^N$
2. Use common randomness to hash $[n] \rightarrow poly(k)$
3. $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$
4. Accept iff $E(x_i, i) = E(y_i, i)$

Main Insight:

\[ \mathbb{E}[G, x \cdot G, y] = \langle x, y \rangle \]

\[ x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \]

Unstated philosophical contribution of CC a la Yao:

Communication with a focus ("only need to determine $f(x, y)$") can be more effective (shorter than $|x|, H(x), H(y), I(x; y)\ldots$)
Many possibilities. Ongoing effort.

Alice + Bob may have estimates of $x$ and $y$.

More generally: $x, y$ correlated.

Knowledge of $f$ – function Bob wants to compute

may not be exactly known to Alice!

Shared randomness

Alice + Bob may not have identical copies.
Part 1: Uncertain Compression
Classical (One-Shot) Compression

- Sender and Receiver have distribution $P \sim [N]$.
- Sender/Receiver agree on Encoder/Decoder $E/D$.
- Sender gets $X \in [N]$; Sends $E(X)$.
- Receiver gets $Y = E(X)$; Decodes $\hat{X} = D(Y)$.
- Requirement: $\hat{X} = X$ (always).
- Performance: $\mathbb{E}_{X \sim P}[|E(X)|]$.

- Trivial Solution: $\mathbb{E}_{X \sim P}[|E(X)|] = \log N$.
- Huffman Coding: Achieves $\mathbb{E}_{X \sim P}[|E(X)|] \leq H(P) + 1$. 

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The (Uncertain Compression) problem

Design encoding/decoding schemes \((E/D)\) s.t.:
- Sender has distribution \(P \sim [N]\)
- Receiver has distribution \(Q \sim [N]\)
- Sender gets \(X \in [N]\); Sends \(E(P,X)\) to receiver.
- Receiver gets \(Y = E(P,X)\); Decodes \(\hat{X} = D(Q,Y)\)
- Want: \(X = \hat{X}\) (provided \(P, Q\) close),

\[\Delta(P, Q) \leq \Delta \text{ if } \left| \log \frac{P(x)}{Q(x)} \right| \leq \Delta \text{ for all } x\]

Motivation: Models natural communication?
Solution (variant of Arith. Coding)

- Uses shared randomness: Sender+Receiver $\leftarrow r \in \{0,1\}^*$
- Use $r$ to define sequences “dictionary”
  - $r_1[1], r_1[2], r_1[3], \ldots$
  - $r_2[1], r_2[2], r_2[3], \ldots$
  - ...
  - $r_N[1], r_N[2], r_N[3], \ldots$

Sender sends prefix of $r_x[1 \ldots L]$ as encoding of $x$

Receiver outputs $\arg\max_z E_z | r_z[1 \ldots L] = r_x[1 \ldots L]$

Want: $L : r_z[1 \ldots L] = r_x[1 \ldots L] \Rightarrow Q(z) < Q(x)$;

$\iff (Q(z) > Q(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$

$\iff (P(z) > 4^{-\Delta}P(x) \Rightarrow r_z[1 \ldots L] \neq r_x[1 \ldots L])$

Analysis:

$E_r[L] = 2\Delta + \log \frac{1}{P(x)}$

$E_{x,r}[L] = 2\Delta + H(P)$

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Implications

- Coding scheme reflects the nature of human communication (extend messages till they feel unambiguous).
- Reflects tension between ambiguity resolution and compression.
  - Larger the ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!
- The “shared randomness” assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
  - This is problematic.
Deterministic Compression: Challenge

- Say Alice and Bob have rankings of $N$ movies.
  - Rankings = bijections $\pi, \sigma : [N] \rightarrow [N]$
  - $\pi(i)$ = rank of $i^{th}$ movie in Alice’s ranking.
- Further suppose they know rankings are close.
  - $\forall i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $O(\log^* n)$
Open Questions (Compression)

- Best Deterministic Uncertain Compression?
  - Best known: \( O(H(P) + \log \log N) \)
  - Dependence on \( N \)?
  - Leading constant?

- Does Private Randomness help?
  - Can we do \( O(H(P) + \log \log \log \log N) \)?

- Movie ranking problem:
  - Dependence on \( N \) necessary?
    - \( \Rightarrow \) Compression length w. Det/Priv. Randomness grows with \( N \)
Part 2: Imperfectly Shared Randomness
Model: Imperfectly Shared Randomness

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where
  
  $(r, s) = \text{i.i.d. sequence of correlated pairs } (r_i, s_i)_i$;
  
  $r_i, s_i \in \{-1, +1\}$; $\mathbb{E}[r_i] = \mathbb{E}[s_i] = 0$; $\mathbb{E}[r_i s_i] = \rho \geq 0$.

- Notation:
  - $\text{isr}_\rho(f) = \text{cc of } f \text{ with } \rho$-correlated bits.
  - $\text{cc}(f)$: Perfectly Shared Randomness cc.
  - $\text{priv}(f)$: cc with PRIVate randomness

- Starting point: for Boolean functions $f$
  - $\text{cc}(f) \leq \text{isr}_\rho(f) \leq \text{priv}(f) \leq \text{cc}(f) + \log n$
  - What if $\text{cc}(f) \ll \log n$? E.g. $\text{cc}(f) = O(1)$
Imperfectly Shared Randomness: Results

- Model first studied by [Bavarian, Gavinsky, Ito’14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $\text{isr}(\text{Equality}) = O(1)$ (among other things)

- Our Results: [Canonne, Guruswami, Meka, S’15]
  - Generally: $\text{cc}(f) \leq k \Rightarrow \text{isr}(f) \leq 2^k$
  - Converse: $\exists f \text{ with } \text{cc}(f) \leq k \& \text{isr}(f) \geq 2^k$
Equality Testing (our proof)

- **Key idea:** Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
  - $x = y \Rightarrow \langle X, Y \rangle = N$
  - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$

- **Estimating inner products:**
  - Building on sketching protocols...
  - Alice: Picks Gaussians $G_1, \ldots, G_t \in \mathbb{R}^N$, Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_\rho(1)$ bits suffice if $G \approx_\rho G'$
General One-Way Communication

- **Idea:** All communication $\leq$ Inner Products
  - (For now: Assume one-way-cc($f$) $\leq k$)
    - For each random string $R$
      - Alice’s message $= i_R \in [2^k]$  
      - Bob’s output $= f_R(i_R)$ where $f_R: [2^k] \rightarrow \{0,1\}$  
      - W.p. $\geq \frac{2}{3}$ over $R$, $f_R(i_R)$ is the right answer.
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- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of $f_R$).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle$; $X = (x_R)_R$; $Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm \epsilon$ with $O_p\left(\frac{1}{\epsilon^2}\right)$ communication.
Two-way communication

- Still decided by inner products.

- Simple lemma:
  - \( \exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k} \) convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of \( \pi_A \in K_A^k, \pi_B \in K_B^k \) equals \( \langle \pi_A, \pi_B \rangle \)

- Putting things together:

Theorem: \( cc(f) \leq k \Rightarrow isr(f) \leq O_\rho(2^k) \)
The Tightness Example

- **Sparse Gap Inner Product:**
  - Alice \( x \leftarrow x \in \{0,1\}^n; \) wt\( (x) \leq 2^{-k} \cdot n \) (Sparse)
  - Bob \( y \leftarrow y \in \{-1,1\}^n; \)
  - Decide: \( \langle x, y \rangle \geq (.9)2^{-k}n \) or \( \langle x, y \rangle \leq 0 \)

- **Shared randomness protocol:**
  - Alice communicates random bit \( i \) s.t. \( x_i = 1 \)
  - Bob outputs \( y_i \)

- **Lower bound:**
  - Invariance principle, Gap Hamming Distance
Open Questions (I.S.R.)

- **Exponential gap for total function?**
  - \( \exists f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}, \) with
    \[ cc(f) \leq k, \quad \& \quad isr - cc(f) \geq 2^k \]

- **Level of correlation?**
  - \( \exists f : \{0,1\}^n \times \{0,1\}^n \to \{0,1,?\}, \) with
    \[ isr_9(f) \leq k, \quad \& \quad isr_1(f) \geq 2^k \]

- **Does interaction help if randomness is not perfectly shared?**
Part 3: Uncertain Functionality
Bob wishes to compute $f(x, y)$; Alice knows $g \approx f$;

Alice, Bob given $g, f$ explicitly. (Input size $\sim 2^n$)

Modelling Questions:

- What is $\approx$?
- Is it reasonable to expect to compute $f(x, y)$? E.g., $f(x, y) = f'(x)$? Can’t compute $f(x, y)$ without communicating $x$

Answers:

- Assume $x, y \sim \{0, 1\}^n \times \{0, 1\}^n$ uniformly.
- $f \approx_\delta g$ if $\delta(f, g) \leq \delta$.
- Suffices to compute $h(x, y)$ for $h \approx_\epsilon f$
Results - 1

- Thm [Ghazi,Komargodski,Kothari,S.]: \( \exists f, g, \mu \) s.t. 
  \( cc_{\mu,1}^{1\text{way}}(f), cc_{\mu,1}^{1\text{way}}(g) = 1 \) and \( \delta_{\mu}(f, g) = o(1) \); but uncertain communication = \( \Omega(\sqrt{n}) \);

- Thm [GKKS]: But not if \( x \perp y \) (in 1-way setting).
  - (2-way, even 2-round, open!)

- Main Idea:
  - Canonical 1-way protocol for \( f \):
    - Alice + Bob share random \( y_1, \ldots, y_m \in \{0,1\}^n \).
    - Alice sends \( f(x, y_1), \ldots, f(x, y_m) \) to Bob.
    - Protocol used previously ... but not as “canonical”.
  - Canonical protocol robust when \( f \approx g \).
Open Questions (Uncertain Functionality)

- What happens when $x, y$ correlated?
  - [Ghazi-S.] ∃functions where cc grows with $I(x; y)$
    - Exact dependence?
    - Is $I(x; y)$ right measure?

- What happens to communication with multiple rounds?
  - Two rounds?

- What is the right task that captures uncertainty in natural communication?
Conclusions

- Positive view of communication complexity: Communication with a focus can be effective!
- Context Important:
  - New layer of uncertainty.
  - New notion of scale (context LARGE)
    - Importance of $o(\log n)$ additive factors.
- Many “uncertain” problems can be solved without resolving the uncertainty (which is a good thing)
- Many open directions+questions
Thank You!