Communication Amid Uncertainty

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Obligatory Sales Pitch

- Most of communication theory [a la Shannon, Hamming]:
  - Built around sender and receiver perfectly synchronized.
- Most Human communication ...
  - ... does not assume perfect synchronization.
- And increasingly ... device-device communication can also not rely on this.
- Can we build a mathematical theory of imperfectly synchronized communication?
  - What are questions/answers?

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Context in Communication

- In most forms of communication:
  - Sender + Receiver share (huuuuge) context
    - In human comm: Language, news, Social
    - In computer comm: Protocols, Codes, Distributions
    - Helps compress communication

- Perfectly shared $\Rightarrow$ Can be abstracted away.
- Imperfectly shared $\Rightarrow$ What is the cost?
  - How to study?
Communication Complexity

The model \((\text{with shared randomness})\)

\[ x \quad f: (x, y) \mapsto \Sigma \quad y \]

\[ R = $$$ \]

Usually studied for lower bounds. This talk: CC as +ve model.

\[ CC(f) = \# \text{ bits exchanged by best protocol} \]

\[ f(x, y) \quad \text{w.p. } 2/3 \]
Aside: Easy CC Problems [Ghazi, Kamath, S’15]

这些问题是否有大输入和小通信?

- 等值测试:
  - $Eq(x, y) = 1 \iff x = y; \quad O(n)$

- 哈明距离:
  - $H_k(x, y) = 1 \iff \Delta(x, y) \leq k$

- 小集合交集:
  - $\cap_k (x, y) = 1 \iff wt(x), wt(y) \leq k$

Unstated philosophical contribution of CC a la Yao:

Communication with a focus ("only need to determine $f(x, y)"") can be more effective (shorter than $|x|, H(x), H(y), I(x; y)\ldots$)

Protocol:
Fix ECC $E: \{0,1\}^n \rightarrow \{0,1\}^N$.

Use common randomness to hash $[n] \rightarrow poly(k)$

$x = (x_1, \ldots, x_n)$

$y = (y_1, \ldots, y_n)$

$\langle x, y \rangle \triangleq \sum_i x_i y_i$
Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of $x$ and $y$.
  - More generally: $x, y$ correlated.
- Knowledge of $f$ – function Bob wants to compute.
  - May not be exactly known to Alice!
- Shared randomness.
  - Alice + Bob may not have identical copies.
Part 1: Uncertain Compression
Classical (One-Shot) Compression

- Sender and Receiver have distribution $P \sim [N]$
- Sender/Receiver agree on Encoder/Decoder $E/D$
- Sender gets $X \in [N]$; Sends $E(X)$
- Receiver gets $Y = E(X)$; Decodes $\hat{X} = D(Y)$
- Requirement: $\hat{X} = X$ (always)
- Performance: $\mathbb{E}_{X \leftarrow P}[|E(X)|]$

- Trivial Solution: $\mathbb{E}_{X \leftarrow P}[|E(X)|] = \log N$
- Huffman Coding: Achieves $\mathbb{E}_{X \leftarrow P}[|E(X)|] \leq H(P) + 1$
The (Uncertain Compression) problem

- Design encoding/decoding schemes \((E/D)\) s.t.:
  - Sender has distribution \(P \sim [N]\)
  - Receiver has distribution \(Q \sim [N]\)
  - Sender gets \(X \in [N]\); Sends \(E(P,X)\) to receiver.
  - Receiver gets \(Y = E(P,X)\); Decodes \(\hat{X} = D(Q,Y)\)
  - Want: \(X = \hat{X}\) (provided \(P, Q\) close),

\[
\Delta(P, Q) \leq \Delta \quad \text{if} \quad \left| \log \frac{P(x)}{Q(x)} \right| \leq \Delta \quad \text{for all} \quad x
\]

- Motivation: Models natural communication?
Solution (variant of Arith. Coding)

- Uses shared randomness: Sender+Receiver $\leftarrow r \in \{0,1\}^*$
- Use $r$ to define sequences “dictionary”
  - $r_1[1], r_1[2], r_1[3], ...$
  - $r_2[1], r_2[2], r_2[3], ...$
  - ...
  - $r_N[1], r_N[2], r_N[3], ...$

  Sender sends prefix of $r_x[1...L]$ as encoding of $x$

  Receiver outputs $\text{argmax} \ E_z | r_z[1...L] = r_x[1...L]$

Want: $L : r_z[1...L] = r_x[1...L] \Rightarrow Q(z) < Q(x)$

\[
\begin{align*}
\mathbb{E}_r[L] &= 2\Delta + \log \frac{1}{P(x)} \\
\mathbb{E}_{x,r}[L] &= 2\Delta + H(P)
\end{align*}
\]
Implications

- Coding scheme reflects the nature of human communication (extend messages till they feel unambiguous).
- Reflects tension between ambiguity resolution and compression.
  - Larger the ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!
- The “shared randomness” assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
    - This is problematic.
Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections $\pi, \sigma : [N] \rightarrow [N]$
  - $\pi(i)$ = rank of $i^{th}$ player in Alice’s ranking.
- Further suppose they know rankings are close.
  - $\forall i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $O(\log^* n)$
Part 2: Imperfectly Shared Randomness
Model: **Imperfectly Shared Randomness**

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where
  - $(r, s) = \text{i.i.d. sequence of correlated pairs } (r_i, s_i)_i$;
  - $r_i, s_i \in \{-1, +1\}$; $\mathbb{E}[r_i] = \mathbb{E}[s_i] = 0$; $\mathbb{E}[r_i s_i] = \rho \geq 0$.

- Notation:
  - $isr_\rho(f) = \text{cc of } f \text{ with } \rho\text{-correlated bits.}$
  - $cc(f): \text{Perfectly Shared Randomness cc.}$
  - $priv(f): \text{cc with PRIVate randomness}$

- Starting point: for Boolean functions $f$
  - $cc(f) \leq isr_\rho(f) \leq priv(f) \leq cc(f) + \log n$
  - What if $cc(f) \ll \log n$? E.g. $cc(f) = O(1)$
Imperfectly Shared Randomness: Results

- Model first studied by [Bavarian,Gavinsky,Ito’14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $\text{isr}(\text{Equality}) = O(1)$ (among other things)

- Our Results: [Canonne,Guruswami,Meka,S’15]
  - Generally: $\text{cc}(f) \leq k \Rightarrow \text{isr}(f) \leq 2^k$
  - Converse: $\exists f$ with $\text{cc}(f) \leq k \& \text{isr}(f) \geq 2^k$
Equality Testing (our proof)

- **Key idea:** Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
  - $x = y \Rightarrow \langle X, Y \rangle = N$
  - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$

- **Estimating inner products:**
  - Building on sketching protocols ...
  - Alice: Picks Gaussians $G_1, \ldots, G_t \in \mathbb{R}^N$
  - Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_{\rho}(1)$ bits suffice if $G \approx_{\rho} G'$

Gaussian Protocol
General One-Way Communication

- Idea: All communication \( \leq \) Inner Products
- (For now: Assume one-way-cc\((f) \leq k\))
  - For each random string \( R \)
    - Alice’s message = \( i_R \in [2^k] \)
    - Bob’s output = \( f_R(i_R) \) where \( f_R: [2^k] \rightarrow \{0,1\} \)
    - W.p. \( \geq \frac{2}{3} \) over \( R \), \( f_R(i_R) \) is the right answer.
General One-Way Communication

- For each random string $R$
  - Alice’s message = $i_R \in [2^k]$
  - Bob’s output = $f_R(i_R)$ where $f_R : [2^k] \rightarrow \{0,1\}$
  - W.p. $\geq \frac{2}{3}$, $f_R(i_R)$ is the right answer.

- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of $f_R$).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle$; $X = (x_R)_R$; $Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm \epsilon$ with $O_{\rho} \left( \frac{1}{\epsilon^2} \right)$ communication.
Two-way communication

- Still decided by inner products.

- Simple lemma:
  - $\exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k}$ convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of $\pi_A \in K_A^k, \pi_B \in K_B^k$ equals $\langle \pi_A, \pi_B \rangle$

- Putting things together:

  \[ \text{Theorem: } cc(f) \leq k \Rightarrow isr(f) \leq O_\rho(2^k) \]
Part 3: Uncertain Functionality
Model

- Bob wishes to compute $f(x, y)$; Alice knows $g \approx f$;
- Alice, Bob given $g, f$ explicitly. (Input size $\sim 2^n$)
- Modelling Questions:
  - What is $\approx$?
  - Is it reasonable to expect to compute $f(x, y)$?
  - E.g., $f(x, y) = f'(x)$? Can’t compute $f(x, y)$ without communicating $x$
- Answers:
  - Assume $x, y \sim \{0,1\}^n \times \{0,1\}^n$ uniformly.
  - $f \approx_\delta g$ if $\delta(f, g) \leq \delta$.
  - Suffices to compute $h(x, y)$ for $h \approx_\epsilon f$
Results - 1

- Thm [Ghazi, Komargodski, Kothari, S.]: \( \exists f, g, \mu \) s.t. \( cc_{\mu,1}^{1\text{way}}(f), cc_{\mu,1}^{1\text{way}}(g) = 1 \) and \( \delta_\mu(f, g) = o(1) \); but uncertain communication = \( \Omega(\sqrt{n}) \);
- Thm [GKKS]: But not if \( x \perp y \) (in 1-way setting).
  - (2-way, even 2-round, open!)
- Main Idea:
  - Canonical 1-way protocol for \( f \):
    - Alice + Bob share random \( y_1, \ldots, y_m \in \{0,1\}^n \).
    - Alice sends \( f(x, y_1), \ldots, f(x, y_m) \) to Bob.
    - Protocol used previously ... but not as “canonical”.
  - Canonical protocol robust when \( f \approx g \).
Conclusions

- Positive view of communication complexity: Communication with a focus can be effective!
- Context Important:
  - New layer of uncertainty.
  - New notion of scale (context LARGE)
    - Importance of $o(\log n)$ additive factors.
- Many “uncertain” problems can be solved without resolving the uncertainty (which is a good thing)
- Many open directions + questions
Thank You!