Communication Amid Uncertainty

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Context in Communication

- Sender + Receiver share (huuuge) context
  - In human comm: Language, news, Social
  - In computer comm: Protocols, Codes, Distributions
  - Helps compress communication

- Perfectly shared $\Rightarrow$ Can be abstracted away.
- Imperfectly shared $\Rightarrow$ What is the cost?
  - How to study?
The model (with shared randomness)

\[ f : (x, y) \mapsto \Sigma \]

\[ R = $$$ \]

Usually studied for lower bounds. This talk: CC as +ve model.

\[ CC(f) = \# \text{ bits exchanged by best protocol} \]

\[ f(x, y) \text{ w.p. } 2/3 \]
Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of $x$ and $y$
  - More generally: $x, y$ correlated.
- Knowledge of $f$ – function Bob wants to compute
  - may not be exactly known to Alice!
- Shared randomness
  - Alice + Bob may not have identical copies.
Part 1: Uncertain Compression
Classical (One-Shot) Compression

- Sender and Receiver have distribution $P \sim [N]$
- Sender/Receiver agree on Encoder/Decoder $E/D$
- Sender gets $X \in [N]$; Sends $E(X)$
- Receiver gets $Y = E(X)$; Decodes $\hat{X} = D(Y)$
- Requirement: $\hat{X} = X$ (always)
- Performance: $\mathbb{E}_{X \sim P}[|E(X)|]$

- Trivial Solution: $\mathbb{E}_{X \sim P}[|E(X)|] = \log N$
- Huffman Coding: Achieves $\mathbb{E}_{X \sim P}[|E(X)|] \leq H(P) + 1$
The (Uncertain Compression) problem

Design encoding/decoding schemes ($E/D$) s.t.:

- Sender has distribution $P \sim [N]$
- Receiver has distribution $Q \sim [N]$
- Sender gets $X \in [N]$; Sends $E(P, X)$ to receiver.
- Receiver gets $Y = E(P, X)$; Decodes $\hat{X} = D(Q, Y)$
- Want: $X = \hat{X}$ (provided $P, Q$ close),

$\Delta(P, Q) \leq \Delta$ if $2^{-\Delta} \leq \frac{\log P(x)}{\log Q(x)} \leq 2^\Delta$ for all $x$

Motivation: Models natural communication?
Solution (variant of Arith. Coding)

- Uses shared randomness: Sender+Receiver $\leftarrow r \in \{0,1\}^*$
- Use $r$ to define sequences “dictionary”
  - $r_1[1], r_1[2], r_1[3], ...$
  - $r_2[1], r_2[2], r_2[3], ...$
  - ...
  - $r_N[1], r_N[2], r_N[3], ...$
- Sender sends prefix of $r_x[1 ... L]$ as encoding of $x$
- Receiver outputs $\text{arg max } Q(z) | r_z[1 ... L] = r_x[1 ... L]$
- Want: $L : r_z[1 ... L] = r_x[1 ... L] \Rightarrow Q(z) < Q(x);$
  $\iff (Q(z) > Q(x) \Rightarrow r_z[1 ... L] \neq r_x[1 ... L])$
  $\iff (P(z) > 4^{-\Delta}P(x) \Rightarrow r_z[1 ... L] \neq r_x[1 ... L])$

Analysis:

$$\mathbb{E}_r[L] = 2\Delta + \log \frac{1}{P(x)}$$
$$\mathbb{E}_{x,r}[L] = 2\Delta + H(P)$$
Implications

- Coding scheme reflects the nature of human communication (extend messages till they feel unambiguous).

- Reflects tension between ambiguity resolution and compression.
  - Larger the ((estimated) gap in context), larger the encoding length.
  - Entropy is still a valid measure!

- The “shared randomness” assumption
  - A convenient starting point for discussion
  - But is dictionary independent of context?
  - This is problematic.
Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections \( \pi, \sigma : [N] \rightarrow [N] \)
  - \( \pi(i) \) = rank of \( i \)th player in Alice’s ranking.
- Further suppose they know rankings are close.
  - \( \forall i \in [N]: |\pi(i) - \sigma(i)| \leq 2 \).
- Bob wants to know: Is \( \pi^{-1}(1) = \sigma^{-1}(1) \)
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – \( O(1) \)
  - Deterministically?
    - With Elad Haramaty: \( O(\log^* n) \)
Part 2: Imperfectly Shared Randomness
Model: Imperfectly Shared Randomness

- Alice $\leftarrow r$; and Bob $\leftarrow s$ where
  \((r, s) = \text{i.i.d. sequence of correlated pairs } (r_i, s_i)_i;\)
  \(r_i, s_i \in \{-1, +1\}; \mathbb{E}[r_i] = \mathbb{E}[s_i] = 0; \mathbb{E}[r_is_i] = \rho \geq 0.\)

- Notation:
  - \(isr_\rho(f) = \text{cc of } f \text{ with } \rho\)-correlated bits.
  - \(cc(f): \text{Perfectly Shared Randomness cc.}\)
  - \(priv(f): \text{cc with PRIVate randomness}\)

- Starting point: for Boolean functions \(f\)
  - \(cc(f) \leq isr_\rho(f) \leq priv(f) \leq cc(f) + \log n\)
  - What if \(cc(f) \ll \log n? \text{E.g.} cc(f) = O(1)\)

\(
\rho \leq \tau \Rightarrow isr_\rho(f) \geq isr_\tau(f)
\)
Imperfectly Shared Randomness: Results

- Model first studied by [Bavarian,Gavinsky,Ito’14] ("Independently and earlier").
  - Their focus: Simultaneous Communication; general models of correlation.
  - They show $isr(\text{Equality}) = O(1)$ (among other things)

- Our Results:[Canonne,Guruswami,Meka,S’15]
  - Generally: $cc(f) \leq k \Rightarrow isr(f) \leq 2^k$
  - Converse: $\exists f \text{ with } cc(f) \leq k \& isr(f) \geq 2^k$
Aside: Easy CC Problems [Ghazi, Kamath, S’15]

- **Equality testing:**
  \[ EQ(x, y) = 1 \iff x = y; \]
  \[ CC(EQ) = O(1) \]

- **Hamming distance:**
  \[ H_k(x, y) = 1 \iff \Delta(x, y) \leq k; \]
  \[ CC(H_k) = O(k \log k) \] [Huang et al.]

- **Small set intersection:**
  \[ \cap_k(x, y) = 1 \iff \text{wt}(x), \text{wt}(y) \leq k \]
  \[ \exists i \in [n], x_i = y_i. \]
  \[ CC(\cap_k) = O(k) \] [Håstad-Wigderson]

**Protocol:**

Fix ECC \[ E : \{0,1\}^N \to \{0,1\}^N \]

**poly(k) Protocol:**

Use common randomness to hash \([n] \to k^2\)

\[ x = (x_1, \ldots, x_n) \]
\[ y = (y_1, \ldots, y_n) \]

\[ \langle x, y \rangle \triangleq \sum_i x_i y_i \]

Unstated philosophical contribution of CC a la Yao:
Communication with a **focus** ("only need to determine \( f(x, y) \)"")
can be more **effective** (shorter than \(|x|, H(x), H(y), I(x; y) \ldots \))
Equality Testing (our proof)

- **Key idea:** Think inner products.
  - Encode $x \mapsto X = E(x); y \mapsto Y = E(y); X, Y \in \{-1, +1\}^N$
  - $x = y \Rightarrow \langle X, Y \rangle = N$
  - $x \neq y \Rightarrow \langle X, Y \rangle \leq N/2$

- **Estimating inner products:**
  - Building on sketching protocols ...
  - Alice: Picks Gaussians $G_1, \ldots, G_t \in \mathbb{R}^N$,
  - Sends $i \in [t]$ maximizing $\langle G_i, X \rangle$ to Bob.
  - Bob: Accepts iff $\langle G'_i, Y \rangle \geq 0$
  - Analysis: $O_\rho(1)$ bits suffice if $G \approx_\rho G'$
General One-Way Communication

- Idea: All communication $\leq$ Inner Products
- (For now: Assume one-way-cc($f$) $\leq k$)
  - For each random string $R$
    - Alice’s message $= i_R \in [2^k]$
    - Bob’s output $= f_R(i_R)$ where $f_R: [2^k] \rightarrow \{0,1\}$
    - W.p. $\geq \frac{2}{3}$ over $R$, $f_R(i_R)$ is the right answer.
General One-Way Communication

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  - Alice’s message = $i_R \in [2^k]$
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- Vector representation:
  - $i_R \mapsto x_R \in \{0,1\}^{2^k}$ (unit coordinate vector)
  - $f_R \mapsto y_R \in \{0,1\}^{2^k}$ (truth table of $f_R$).
  - $f_R(i_R) = \langle x_R, y_R \rangle$; Acc. Prob. $\propto \langle X, Y \rangle$; $X = (x_R)_R$; $Y = (y_R)_R$
  - Gaussian protocol estimates inner products of unit vectors to within $\pm \varepsilon$ with $O_\rho \left( \frac{1}{\varepsilon^2} \right)$ communication.
Two-way communication

- Still decided by inner products.

- Simple lemma:
  - $\exists K_A^k, K_B^k \subseteq \mathbb{R}^{2^k}$ convex, that describe private coin k-bit comm. strategies for Alice, Bob s.t. accept prob. of $\pi_A \in K_A^k, \pi_B \in K_B^k$ equals $\langle \pi_A, \pi_B \rangle$

- Putting things together:

  Theorem: $cc(f) \leq k \Rightarrow isr(f) \leq O(2^k)$
Part 3: Uncertain Functionality
Model

- Bob wishes to compute $f(x, y)$; Alice knows $g \approx f$;
- Alice, Bob given $g, f$ explicitly. (Input size $\sim 2^n$)

**Modelling Questions:**
- What is $\approx$?
- Is it reasonable to expect to compute $f(x, y)$?
  - E.g., $f(x, y) = f'(x)$? Can’t compute $f(x, y)$ without communicating $x$

**Answers:**
- Assume $x, y \sim \{0,1\}^n \times \{0,1\}^n$ uniformly.
- $f \approx_\delta g$ if $\delta(f, g) \leq \delta$.
- Suffices to compute $h(x, y)$ for $h \approx_\epsilon f$
Results - 1

- Thm [Ghazi, Komargodski, Kothari, S.]: \( \exists f, g, \mu \) s.t. \( cc_{\mu,1}^{1\text{way}}(f), cc_{\mu,1}^{1\text{way}}(g) = 1 \) and \( \delta_\mu(f, g) = o(1) \); but uncertain communication = \( \Omega(\sqrt{n}) \);

- Thm [GKKS]: But not if \( x \perp y \) (in 1-way setting).
  - (2-way, even 2-round, open!)

- Main Idea:
  - Canonical 1-way protocol for \( f \):
    - Alice + Bob share random \( y_1, \ldots, y_m \in \{0,1\}^n \).
    - Alice sends \( f(x, y_1), \ldots, f(x, y_m) \) to Bob.
    - Protocol used previously ... but not as "canonical".
  - Canonical protocol robust when \( f \approx g \).
Conclusions

- Positive view of communication complexity: Communication with a focus can be effective!
- Context Important:
  - New layer of uncertainty.
  - New notion of scale (context LARGE)
    - Importance of $o(\log n)$ additive factors.
- Many “uncertain” problems can be solved without resolving the uncertainty (which is a good thing)
- Many open directions+questions
Thank You!