# Locality in Coding Theory

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# Locality ≠ LOCALity

- Local ≡ "Influenced by small neighborhood"
- Two views in literature:
  - Focus on "small"
    - Leads to greater efficiency in algorithms, but not necessarily structure.
  - Focus on "neighborhood"
    - Suitable in distributed computing.
    - More structure in algorithms
    - Input to algorithm = architecture for solution
- Coding Theory: Focus on "small"

# Locality in Coding Theory

- Historical accident?
- Classical concepts:
  - Local decodability
  - Local list-decodability
  - Local testability
- Striking recent developments:
- Modern Concepts:
  - Local Reconstruction Codes
  - Some constructions

# (Classical) Challenge of Coding Theory

- Protect data from (inevitable) error
- Question: Does ability to protect go ↑ with size ↑
  - Shannon's surprising answer (surprise in 1948): YES
  - Cost = ? Recovery time ↑ with size ↑
- Current options to store massive data:
  - Break data into small chunks.
    - Good recovery time but lose twice in reliability: (1) Reliability per chunk; (2) Union bound on chunks.
  - Keep data in one big chunk:
    - Good reliability, but poor recovery time.
- Locality (if achievable): Best of both worlds.
  - Recovery time = sublinear?

# Basic Notions:

- Code: $E: \Sigma^k \to \Sigma^n$; $\delta(E) = \min_{x \neq y}\{\delta(E(x), E(y))\}$
  - Testing: Given $E, r \in \Sigma^n$; $\exists m$ s.t. $r = E(m)$?
  - Decoding: Given $E, r$ find $m$ s.t. $\delta(r, E(m)) \leq \delta_0$
- Locality: Do above in sublinear time!
  - Local testing:
    - Make few ($\ell$) queries to $r$;
    - Reject with prob. $\geq \epsilon \cdot \delta(r, E)$
  - Locally Decodable (correctible) Code:
    - Given $E, r, i$ make few ($\ell$) queries to $r$; output $E(m)_i$
  - Local Reconstruction Codes:
    - LDCs for $\delta_0 = \frac{1}{n}$; (even simpler: erasures only!)

# Progress Report: LTCs

- Motivations:
  - Entirely mathematical?
    - PCPs … XOR lemma, extremal graphs, …
- Initial belief: Locality $c(\epsilon) \to \infty$ as $n = k^{1+\epsilon}$; $\epsilon \to 0$
  - Disproved many times:
    - [GS,BGHSV,BS,Dinur,Meir]: $c = 3$ with $n = k \cdot \mathrm{polylog}\, k$
- Semi-recent belief: Locality $\ell = k^{\epsilon} \Rightarrow \frac{k}{n} \to 0$ as $\epsilon \to 0$
  - Disproved fewer times:
    - [Viderman'11]: $\ell = k^{\epsilon}$ and $\frac{k}{n} = 1 - \delta$, $\forall \epsilon, \delta > 0$.
    - [Kopparty,Meir,Ron-Zewi,Saraf'16]: $\ell = \widetilde{\mathrm{polylog}}(k)$ achievable for "free"
- Practical use?

# Progress Report: LDCs

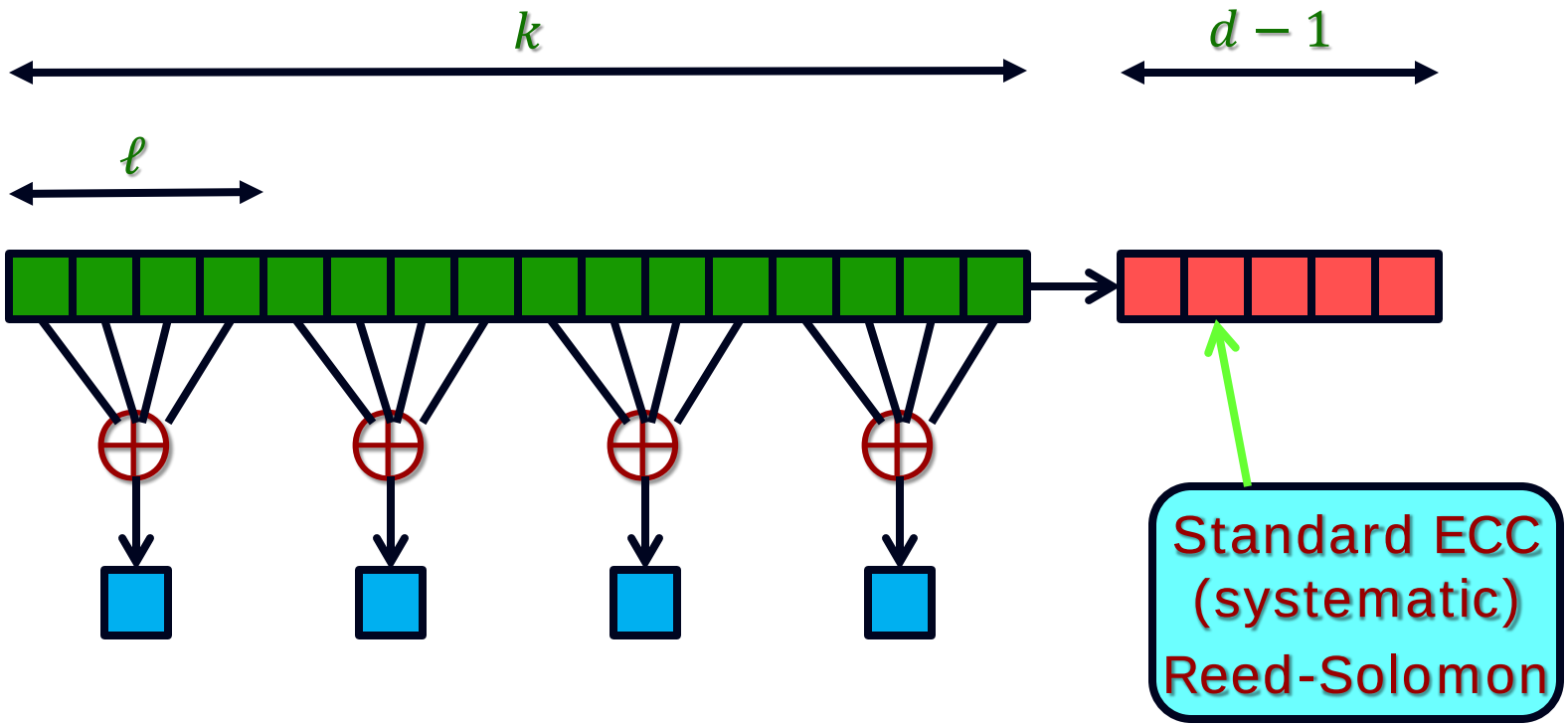
- Motivations:
  - Mathematical … + (in hindsight) Practical?
- Initial belief: Locality $\ell(\epsilon) \to \infty$ as $n = 2^{k^\epsilon}$; $\epsilon \to 0$
  - Disproved (2006-9) [Yekhanin, Raghavendra, Efremenko]:
    $\ell = 3$; $n = \exp\left(\exp\left(\sqrt{\log k}\right)\right)$
- Semi-recent belief: Locality $\ell = k^\epsilon \Rightarrow \frac{k}{n} \to 0$ as $\epsilon \to 0$
  - Disproved "many times" (2010-2016):
    - [Kopparty,Meir,Ron-Zewi,Saraf'16]: $\ell = 2^{\sqrt{\log k}}$ achievable for "free"
- Practical Use?!

# Local Reconstruction Codes

- Practically motivated notion:
  - Fix small corruption quickly (locally).
  - Remain resilient to many errors.

- $(\ell, d)$-LRC [Gopalan-Huang-Simitci-Yekhanin]:
  - Code of distance $d/n$ s.t. <u>single</u> symbol <u>erasure</u> can be recovered $\ell$-locally.
    - Symbol of message? (Weak Defn.)
    - Symbol of codeword? (Strong Defn.)

# An Easy Construction




Satisfies weak defn.

$$n = k + \frac{k}{\ell} + d - 1 \quad -1$$

Can save ■ if it is $\oplus$ of all ■

# A limiting result

- Recall Singleton Bound: $n \geq k + d - 1$
  - Project Code to first $k-1$ coordinates.
  - Pigeonhole Principle $\Rightarrow$

$$\exists\, x \neq y \text{ s.t. } E(x)_{1:k-1} = E(y)_{1:k-1}$$

- Simple $(\ell, d)$-LRC weaker by $\approx + \frac{k}{\ell}$
- Thm [GHSY]: This is necessary (up to $\pm 1$)
  - Find $k - 1 + (k-1)/\ell$ locations of rank $k-1$ greedily.
  - Project to these coordinates + PHP.

# A "state-of-art" construction

- [Tamo-Barg]: $n = k + \frac{k}{\ell} + d\ (\pm 1)$ (Strong Defn.)
- Code = Subcode of Reed-Solomon Code:
  - Work with $\mathbb{F}_q$ where $r = \ell + 1$ divides $q$; $\omega^r = 1$
  - $\left\{ \left( f(\alpha) \middle| \alpha \in \mathbb{F}_q^* \right) \middle| f(x) = \sum_{\{i \in [k-1]; i \bmod r \neq -1\}} m_i x^i \right\}$
  - $(\ell = 2: p(x) = a + bx + \quad cx^3 + dx^4 + \quad ex^6 + \cdots)$
  - Local groups: $S_a = \{a, a\omega, a\omega^2, \ldots, a\omega^\ell\}$; $r$ points
  - $f|_{S_a} \equiv f(x) \bmod (x^r - a^r)$; deg. $\leq r - 2$!
  - $\left( p(x) \bmod \left(x^3 - 1\right) = (a + c + e \ \ldots) + (b + d \ldots)\, x \right)$

# Conclusions

- Locality in coding theory
  - Diverse collection of notions
  - New problems and solutions (and techniques).
  - Diverse applications.
  - Many open questions!