Mathematical Theories of Communication: Old and New

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Communication = What?

- Today: Digital Communication
  - i.e., Communicating bits/bytes/data …
    - As opposed to “radio waves for sound”

- Challenge? Communication can be …
  - … expensive: (e.g., satellite with bounded energy to communicate)
  - … noisy: (bits flipped, DVD scratched)
  - … interactive: (complicates above further)
  - … contextual (assumes you are running specific OS, IPvX)

Theory = Why?

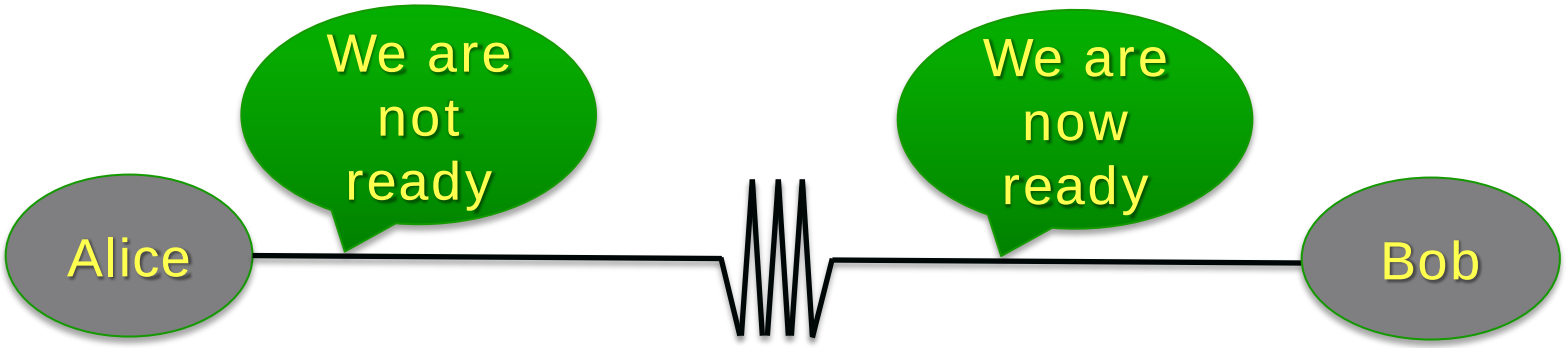
- Why build theory and not just use whatever works?
  - Ad-hoc solutions work today, but will they work tomorrow?
  - Formulating problem allows comparison of solutions!
    - And some creative solutions might be surprisingly better than naïve!
  - Better understanding of limits.
    - What can not be achieved ...

Old? New?

- Why new theories? Were old ones not good enough?
  - Quite the opposite: Old ones were too good.
  - They provided right framework and took us from ground level to “orbit”!
  - And now we can explore all of “space” … but new possibilities leads to new challenges.

Reliable Communication?

- Problem from the 1940s: Advent of digital age.



- Communication media are always noisy
  - But digital information less tolerant to noise!

Reliability by Repetition

- Can repeat (every letter of) message to improve reliability:

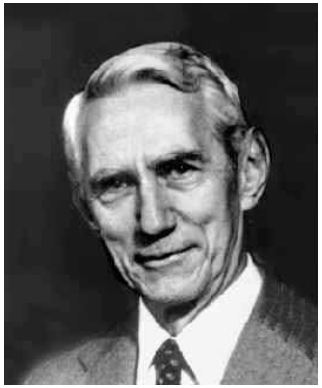
  WWW EEE AAA RRR EEE NNN OOO WWW ...

  ↓

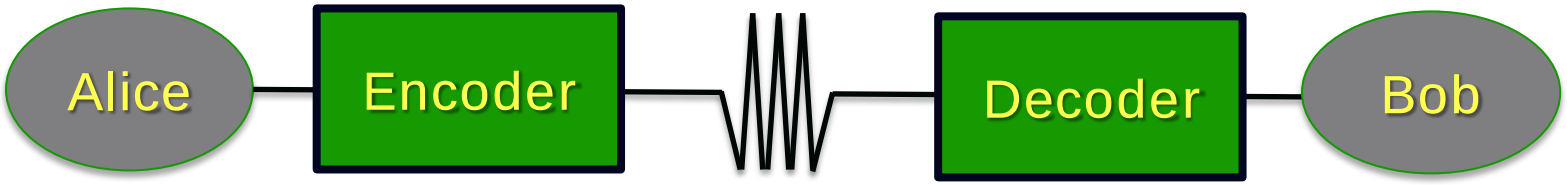
  WXW EEA ARA SSR EEE NMN OOP WWW ...

- Elementary Reasoning:
  - ↑ repetitions ⇒ ↓ Prob. decoding error; but still +ve
  - ↑ length of transmission ⇒ ↑ expected # errors.
  - Combining above: Rate of repetition coding → 0 as length of transmission increases.
- Belief (pre1940):
  - Rate of any scheme → 0 as length → ∞

Shannon's Theory [1948]



- Sender "Encodes" before transmitting
- Receiver "Decodes" after receiving



- Encoder/Decoder arbitrary functions.

$$E: \{0,1\}^k \to \{0,1\}^n$$
$$D: \{0,1\}^n \to \{0,1\}^k$$

  - Rate $= \frac{k}{n}$;
  - Requirement: $m = D(E(m) + \text{error})$ w. high prob.
- What are the best E, D (with highest Rate)?

Shannon's Theorem

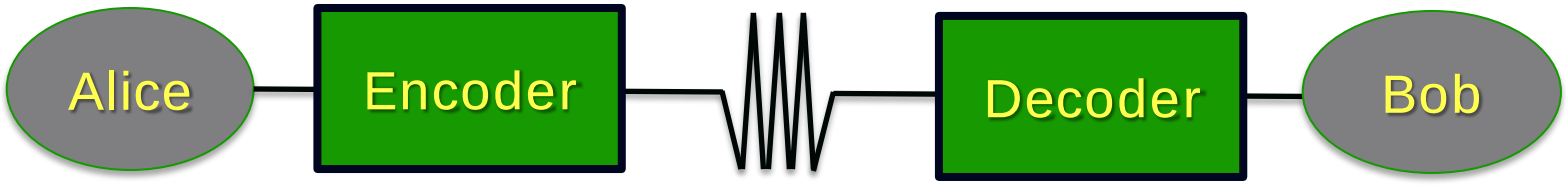
- If every bit is flipped with probability p
  - Rate $\to 1 - H(p)$ can be achieved.

$$H(p) \triangleq p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p}$$

  - This is best possible.
  - Examples:
    - $p = 0 \Rightarrow Rate = 1$
    - $p = \frac{1}{2} \Rightarrow Rate = 0$
    - Monotone decreasing for $p \in (0, \frac{1}{2})$
    - Positive rate for $p = 0.4999$; even if $k \to \infty$

Shannon's contributions

- Far-reaching architecture:

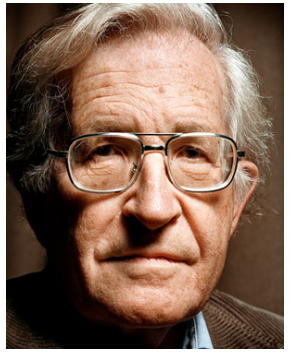


- Profound analysis:
  - First (?) use of probabilistic method.
- Deep Mathematical Discoveries:
  - Entropy, Information, Bit?

Challenges post-Shannon

- Encoding/Decoding functions not "constructive".
  - Shannon picked E at random, D brute force.
  - Consequence:
    - D takes time $\sim 2^k$ to compute (on a computer).
    - E takes time 2^{2^k} to find!
- Algorithmic challenge:
  - Find E, D more explicitly.
  - Both should take time $\sim k, k^2, k^3$ … to compute
  - Solutions: 1948-2017

Chomsky: Theory of "Language"



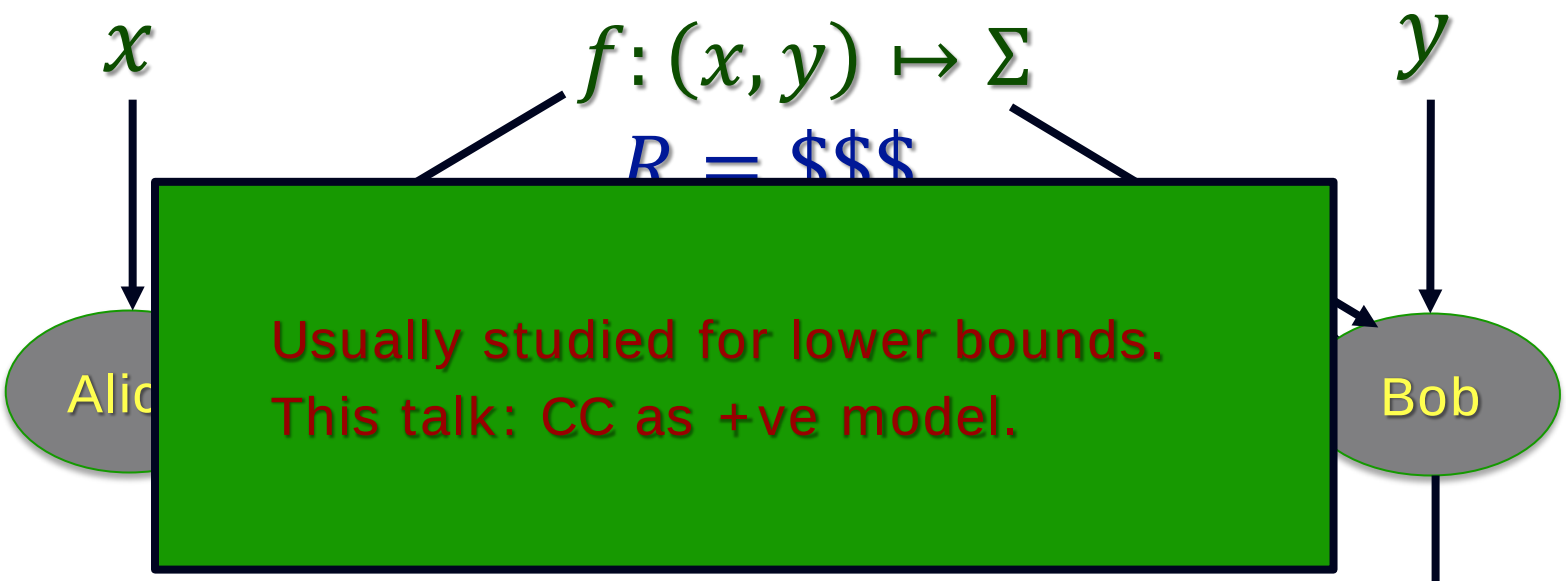
- Formalized syntax in languages mathematically (with caveats).
  - Initially goal was to understand inter-human communication!
    - Structure behind languages
    - Implication on acquisition.
  - Major side-effect: Use of "context-free grammars'' to specify programming languages!
    - Automated parsing, compiling, interpretation!

Modern Theories:

- Communication is interactive
  - E.g., “buying airline ticket online”
- Involve large inputs from either side:
  - Travel agent = vast portfolio of airlines+flights
  - Traveler = complex collection of constraints and preferences.
  - Best protocol ≠ Travel agent sends brochure.
    ≠ Traveller sends entire list of constraints.
- How to model? What happens if errors happen? How well should context be shared?

Communication Complexity: Yao

The model (with shared randomness)



$CC(f)$ = # bits exchanged by best protocol

$f(x,y)$ w.p. 2/3

Some short protocols!

- Problem 1: Alice $\leftarrow x_1, \ldots, x_n$; Bob $\leftarrow y_1, \ldots, y_n$;
- Want to know: $(x_1 - y_1) + \cdots + (x_n - y_n)$
- Solution: Alice $\rightarrow$ Bob: $S \stackrel{\text{def}}{=} x_1 + \cdots x_n$

  Bob $\rightarrow$ Alice: $S - (y_1 + \cdots + y_n)$
- Problem 2: Alice $\leftarrow x_1, \ldots, x_n$; Bob $\leftarrow y_1, \ldots, y_n$;
- Want to know: $(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$
- Solution?
  - Deterministically: Needs n bits of communication.
  - Randomized: Say Alice+Bob $\leftarrow r_1, r_2, \ldots, r_n \in \{-1.+1\}$ random.
  - Alice$\rightarrow$Bob: $S_1 = x_1^2 + \cdots + x_n^2$; $S_2 = r_1 x_1 + \cdots r_n x_n$
  - Bob$\rightarrow$Alice: $T_1 = y + \cdots + y$; $T_2 = r_1 y_1 + \cdots r_n y_n$
  - Thm: Exp $[S_1 + T_1 - 2S_2 T_2] = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$

Application to Buying Air Tickets

- If we can express every flight and every user's preference as n numbers
  - (commonly done in Machine Learning)
  - Then #bits communicated $\approx 2.$ description of final itinerary.
  - Only two rounds of communication!

- Challenge: Express user preferences as numbers!
  - Not yet there … but soon your cellphones will do it!

Interaction + Errors: Schulman

- Consider distributed update of shared document.

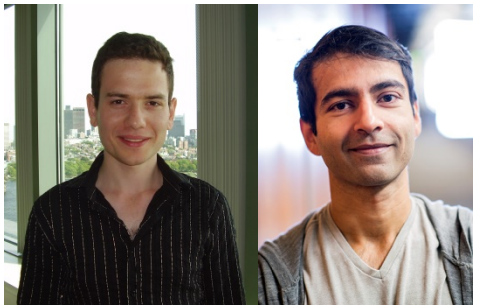
Typical interaction:
Server → User: Current state of document
User → Server: Update

- What if there are errors in interaction?
- Error must be detected immediately?
  - Or else all future communication wasted.
- But too early detection might lead to false alarms!

Interactive Coding Schemes

- If bits are flipped with probability p what is the rate of communication?
- Limits still not precisely determined! But linear for $p \leq \frac{1}{8}$ (scales more like $1 - \sqrt{H(p)}$)

Schulman



Braverman-Rao

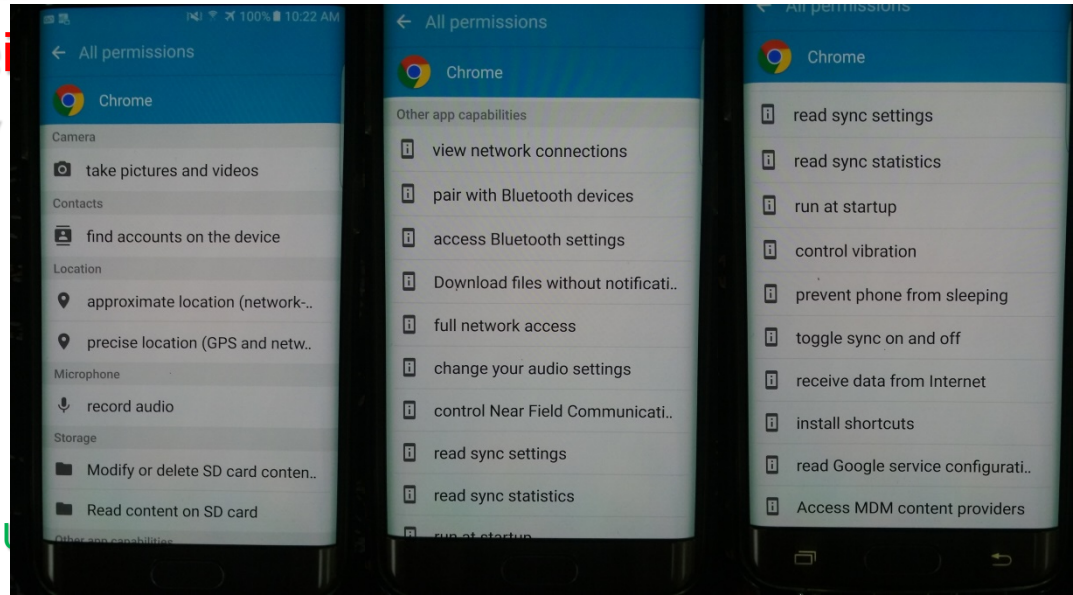
Kol-Raz

Haeupler

- Non-trivial mathematics! Some still not fully constructive
- … surprising even given Shannon theory!

Sales Pitch + Intro

- Most of communication theory [a la Shannon, Hamming]:
  - Built around sender and receiver perfectly synchronized.
  - So large context (codes, protocols, priors) ignored.
- Most Human communication (also device-device)
  - … does not assume perfect synchronization.
  - So context is relevant:
    - Qualitatively (recei
    - and Quantitatively



Aside: Contextual Proofs & Uncertainty

- Mathematical proofs assume large context.
  - "By some estimates a proof that 2+2=4 in ZFC would require about 20000 steps ... so we will use a huge set of axioms to shorten our proofs – namely, everything from high-school mathematics" [Lehman,Leighton,Meyer]
- Context shortens proofs. But context is uncertain!
  - What is "high school mathematics"
    - Is it a fixed set of axioms?
    - Or a set from which others can be derived?
      - Is the latter amenable to efficient reasoning?
      - What is efficiency with large context?

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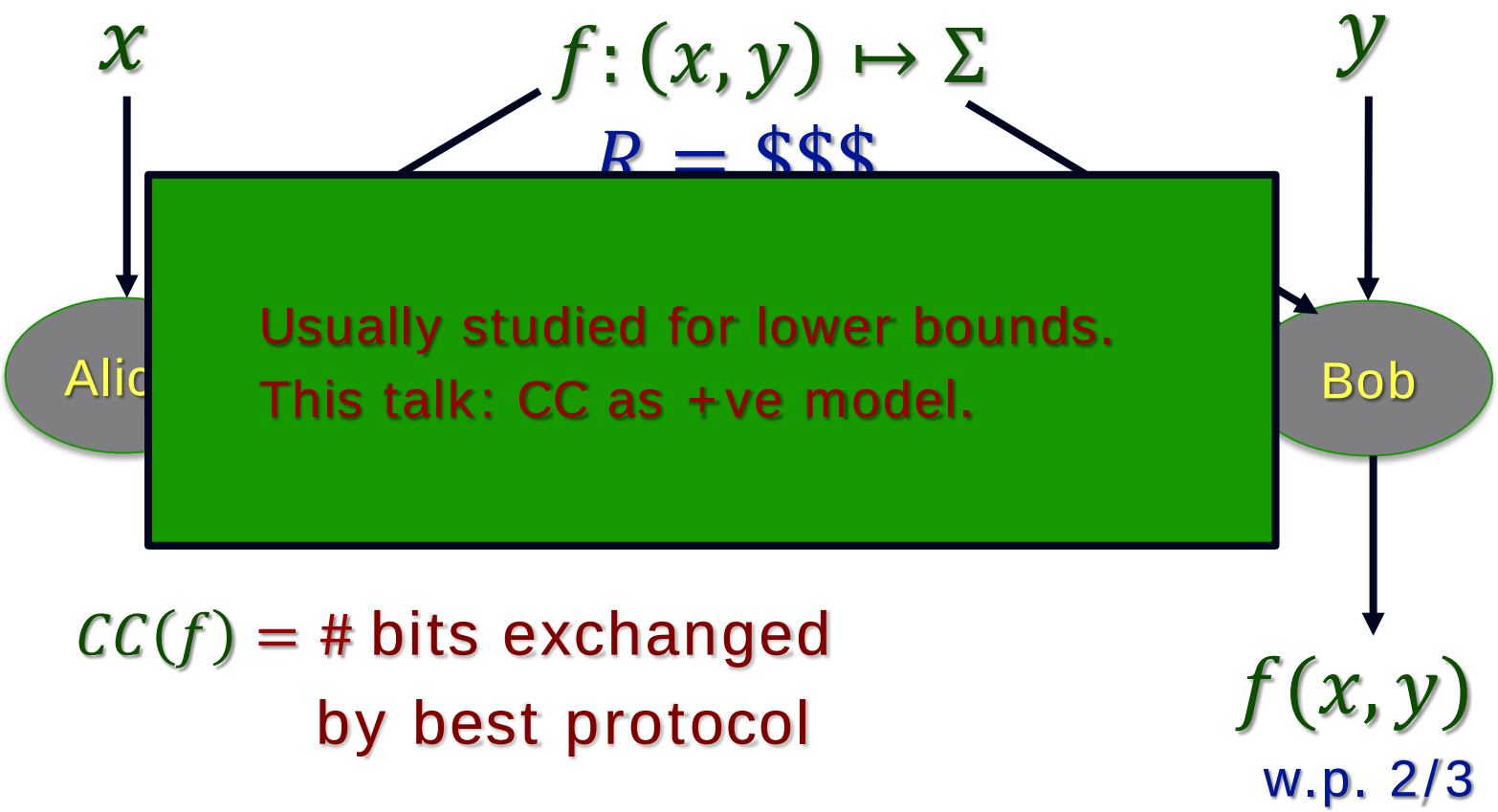
- Most Human communication (also device-device)
  - … does not assume perfect synchronization.
  - So context is relevant:
    - Qualitatively (receiver takes wrong action)
    - and Quantitatively (inputs are long!!)
- Theory? What are the problems?
  - Starting point = Shannon? Yao?

Communication Complexity

The model (with shared randomness)



$CC(f)$ = # bits exchanged by best protocol

Aside: Easy CC Problems [Ghazi,Kamath,S'15]

∃ Problems with large inputs and small communication?

- Equality testing:
  - $EQ(x,y) = 1 \Leftrightarrow x = y$;
- Hamming distance:
  - $H_k(x,y) = 1 \Leftrightarrow \Delta(x,y) \leq k$;
- Small set intersection:
  - $\cap_k (x,y) = 1 \Leftrightarrow$ wt(x), wt$(y$
  - $CC(\cap_k) = O(k)$ [Håstad Wi

Protocol:

$poly(k)$ Proto

Use common

to hash $[n] \rightarrow$

$x = (x_1, \dots, x_n)$

$y = (y_1, \dots, y_n)$

$\langle x, y \rangle \triangleq \sum_i x_i y_i$

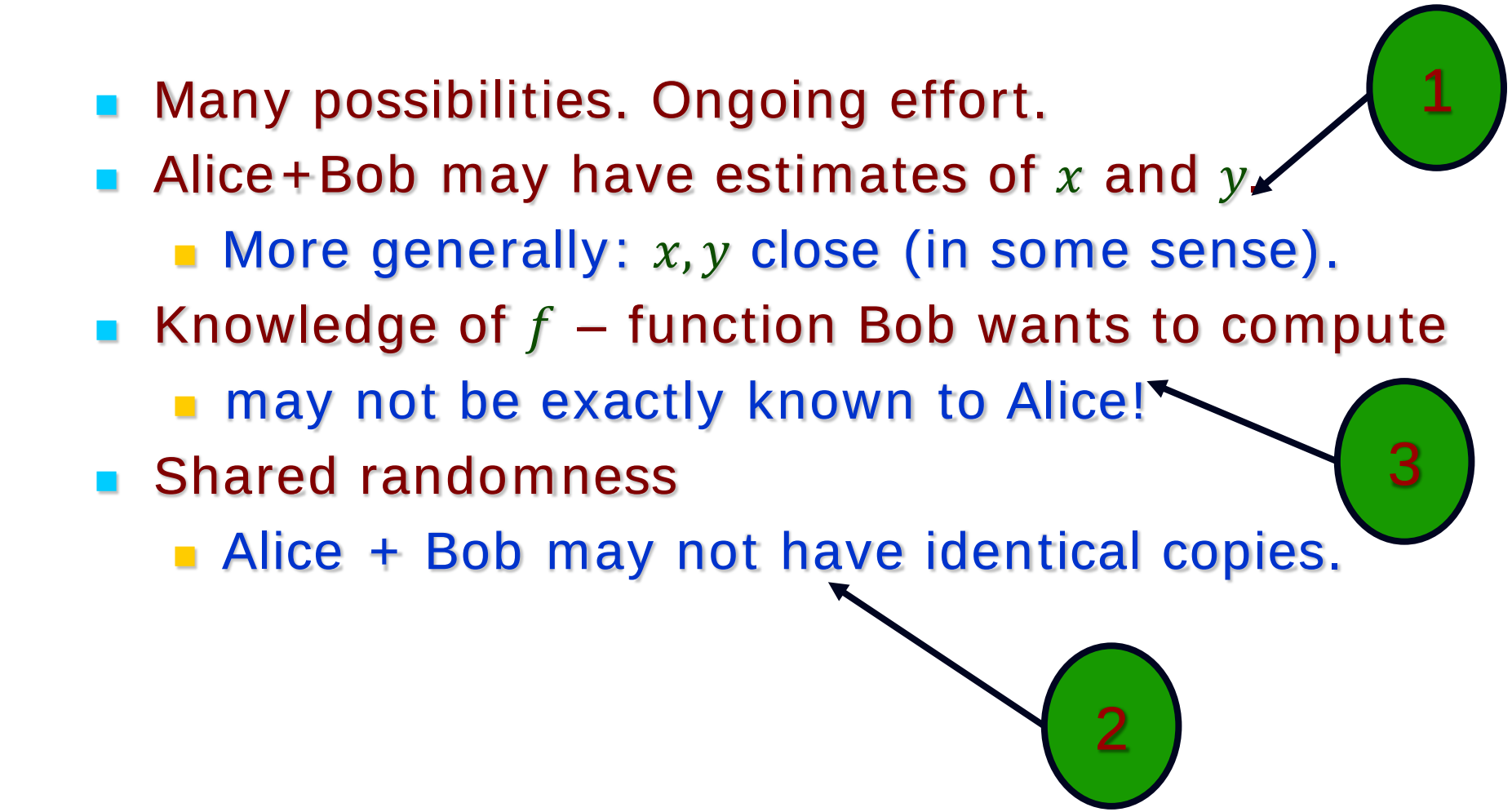
Unstated philosophical contribution of CC a la Yao:

Communication with a focus ("only need to determine $f(x,y)$")

can be more effective (shorter than $|x|, H(x), H(y), I(x;y) \dots$)

Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of x and y.
  - More generally: x, y close (in some sense).
- Knowledge of f – function Bob wants to compute
  - may not be exactly known to Alice!
- Shared randomness
  - Alice + Bob may not have identical copies.



1. Compression

- Classical compression: Alice $\leftarrow P, m \sim P$; Bob $\leftarrow P$;
  - Alice $\rightarrow$ Bob: $y = E_P(m)$; Bob $\hat{m} = D_P(y) \stackrel{?}{=} m$;
  - [Shannon]: $\hat{m} = m$; w. $\mathbb{E}_{m \sim P}[|E_P(m)|] \leq H(P) + 1$
    $H(P) \stackrel{\text{def}}{=} \mathbb{E}_{m \sim P}[-\log P(m)]$
- Uncertain compression [Juba,Kalai,Khanna,S.]
  - Alice $\leftarrow P, m \sim P$; Bob $\leftarrow Q$;
  - Alice $\rightarrow$ Bob: $y = E_P(m)$; Bob $\hat{m} = D_Q(y) \stackrel{?}{=} m$;
  - P, Q Δ-close: $\forall m \; |\log P(m) - \log Q(m)| \leq \Delta$
  - Can we get $\mathbb{E}_{m \sim P}[|E_P(m)|] \leq O(H(P) + \Delta)$?
  - [JKKS] – Yes – with shared randomness.
  - [CGMS] – Yes – with shared correlation.
  - [Haramaty+S.] – Deterministically $O(H(P) + \log\log|\Omega|)$

Universe

Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections $\pi, \sigma : [N] \to [N]$
  - $\pi(i)$ = rank of i^{th} player in Alice's ranking.
- Further suppose they know rankings are close.
  - $\forall\, i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $\tilde{O}(\log^* n)$

Compression as a proxy for language

- Information theoretic study of language?
- Goal of language: Effective means of expressing information/action.
- Implicit objective of language: Make frequent messages short. Compression!
- Frequency = Known globally? Learned locally?
  - If latter – every one can't possibly agree on it;
  - Yet need to agree on language (mostly)!
  - Similar to problem of Uncertain Compression.
  - Studied formally in [Ghazi,Haramaty,Kamath,S. ITCS 17]

2. Imperfectly Shared Randomness

- Recall: Communication becomes more effective with randomness.
  - Identity, Hamming Distance, Small Set Intersection, Inner Product.
- How does performance degrade if players only share correlated variables:
  - E.g. Alice $\leftarrow r$; Bob $\leftarrow s$. $(r,s) = (r_i,t_i)_i$ i.i.d.

$r_i, s_i \in \{-1,1\}$; $\mathbb{E}[r_i] = \mathbb{E}[s_i] = 0$; $\mathbb{E}[r_i s_i] = \rho \in (0,1)$;

- [CGMS '16]:
  - Comm. With perfect randomness = k

    $\Rightarrow$ Comm. With imperfect randomness = $O_\rho(2^k)$

Imperfectly Shared Randomness

- Easy (Complete) Problem:
  - Gap Inner Product: $x, y \in \mathbb{R}^n$
  - $GIP_{c,s}(x, y) = 1$ if $\langle x, y \rangle \geq \epsilon \cdot |x|_2 \cdot |y|_2$;
    $= 0$ if $\langle x, y \rangle \leq 0$
  - Decidable with $O_\rho\left(\frac{1}{\epsilon^2}\right)$ (o.w.) communication
- Hard Problem:
  - Sparse Gap Inner Product: GIP on sparse x
    - $x \in \{0,1\}^n, y \in \{-1,1\}^n$; $|x|_1 = 2\epsilon n$
  - Classical communication = $O\left(\log\frac{1}{\epsilon}\right)$ [uses sparsity]
  - No way to use sparsity with imperfect randomness.

3. Functional Uncertainty

- [Ghazi,Komargodski,Kothari,S. '16]
- Recall positive message of Yao's model:
  - Communication can be brief, if Alice knows what function $f(x,y)$ Bob wants to compute.
- What if Alice only knows f approximately?
  - Can communication still be short?

The Model

- Recall Distributional Complexity:
  - $(x,y) \sim \mu$; $\text{error}_{\mu}(\Pi) \stackrel{\text{def}}{=} \Pr_{x,y\sim\mu}[f(x,y) \neq \Pi(x,y)]$
  - Complexity: $cc_{\mu,\epsilon}(f) \stackrel{\text{def}}{=} \min_{\Pi: error_{\mu}(\Pi)\leq\epsilon} \left\{\max_{x,y}\{|\Pi(x,y)|\}\right\}$
- Functional Uncertainty Model - I:
  - Adversary picks (f,g). Nature picks $(x,y) \sim \mu$
  - Alice $\leftarrow (f,x)$; Bob $\leftarrow (g,y)$; Compute $g(x,y)$
  - Promise: $\delta_{\mu}(f,g) \stackrel{\text{def}}{=} \Pr_{\mu}[f(x,y) \neq g(x,y)] \leq \delta_0$
  - Goal: Compute (any) $\Pi(x,y)$ with $\delta_{\mu}(g,\Pi) \leq \epsilon_1$
    - (just want $\epsilon_1 \to 0$ as $\delta_0 \to 0$)
- If (f,g) part of input; this is complexity of what?

Modelling Uncertainty

- Modelled by graph $\mathcal{G}$ of possible inputs
- Protocols know $\mathcal{G}$ but not (f, g)
- $cc_{\mu,\epsilon}(\mathcal{G}) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \{cc_{\mu,\epsilon}(g)\}$
- $\delta_\mu(\mathcal{G}) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \{\delta_\mu(f,g)\}$
- Uncertain error:

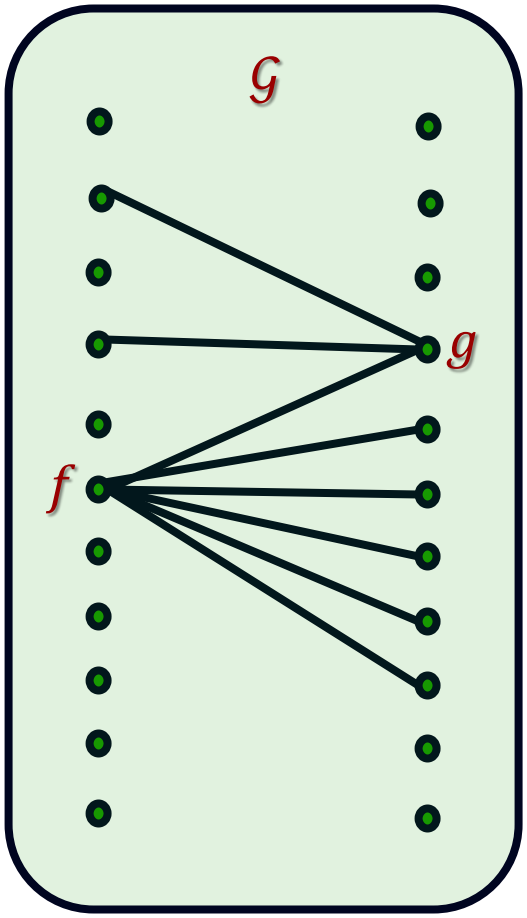
$$error_{\mu,\mathcal{G}}(\Pi) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \left\{\Pr_{\mu}[g(x,y) \neq \Pi(f,g,x,y)]\right\}$$

- Uncertain complexity:

$$Ucc_{\mu,\epsilon}(\mathcal{G}) \stackrel{\text{def}}{=} \min_{\Pi:\text{error}(\Pi)\leq\epsilon} \left\{\max_{f,g,x,y} \{|\Pi(f,g,x,y)|\}\right\}$$

- Compare $cc_{\mu,\epsilon_0}(\mathcal{G})$ vs. $Ucc_{\mu,\epsilon_1}(\mathcal{G})$

  want $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$



Main Results

- Thm 1: (-ve) $\exists \mathcal{G}, \mu$ s.t. $\delta_\mu(\mathcal{G}) = o(1); cc_{\mu,o(1)}(\mathcal{G}) = 1;$ but $Ucc_{\mu,.1}(\mathcal{G}) = \Omega(\sqrt{n})$; ($n = |x| = |y|$)
- Thm 2: (+ve) $\forall \mathcal{G}$, product μ,

$$Ucc^{oneway}_{\mu,\epsilon_1}(\mathcal{G}) = O\left(cc^{oneway}_{\mu,\epsilon_0}(\mathcal{G})\right)$$

where $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$

Protocols are not continuous wrt the function being computed

- Thm 2': (+ve) $\forall \mathcal{G}, \mu$,

$$Ucc^{oneway}_{\mu,\epsilon_1}(\mathcal{G}) = O\left(cc^{oneway}_{\mu,\epsilon_0}(\mathcal{G}) \cdot (1 + I(x;y))\right)$$

where $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$

and $I(x;y)$ = Mutual Information between $x; y$

Details of Negative Result

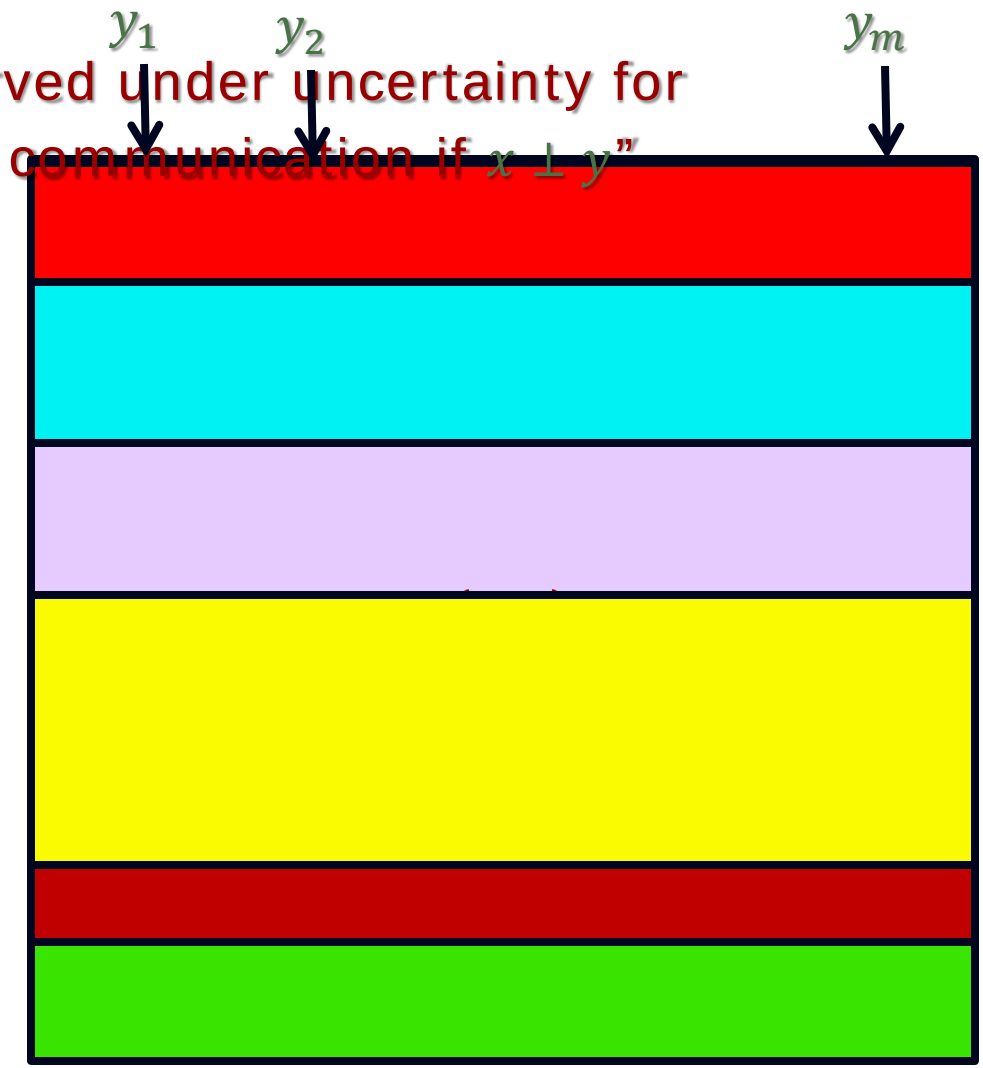
- μ: $x \sim U(\{0,1\}^n); y = Noisy(x)$; $\Pr[x_i \neq y_i] = 1/\sqrt{n}$;
- $\mathcal{G} = \{(\oplus_S (x \oplus y), \oplus_T (x \oplus y)) |\ |S \oplus T| = o(\sqrt{n})\}$
  - $\oplus_S (z) = \oplus_{i \in S} z_i$
- Certain Comm: Alice → Bob: $\oplus_T (x)$
- $\delta_\mu(\mathcal{G}) = \max_{S,T} \left\{ \Pr_{x,y} [\oplus_S (x \oplus y) \neq \oplus_T (x \oplus y)] \right\}$

  $= \max_{U:|U|=o(\sqrt{n})} \left\{ \Pr_{z \sim \text{Bernoulli}\left(\frac{1}{\sqrt{n}}\right)^n} [\oplus_U (z) = 1] \right\} = o(1)$
- Uncertain Lower bound:
  - Standard $cc_{\mu,\epsilon}(F)$ where $F\big((S,x);(T,y)\big) = \oplus_T (x \oplus y)$;
  - Lower bound obtain by picking (S,T) randomly:
    - S uniform; T noisy copy of S

Positive result (Thm. 2)

- Consider comm. Matrix
- Protocol for g partitions matrix in 2^k blocks
- Bob wants to know which block?
- Common randomness: $y_1, \dots, y_m$
- Alice → Bob: $f(x, y_1) \dots f(x, y_m)$
- Bob (whp) recognizes block and uses it.
- $m = O(k)$ suffices.

"CC preserved under uncertainty for one-way communication if $x \perp y$"



Analysis Details

1. W.p. $1-\sqrt{\epsilon}$, $\exists j$ s.t. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \leq \sqrt{\epsilon}$
   - Main idea: If $\Pi_g(x) = \Pi_g(x_j)$ then w.h.p. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \leq \sqrt{\epsilon}$
2. If $j \in [K]$ s.t. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \geq 2\sqrt{\epsilon}$ then $\Pr[j \text{ is selected}] = \exp(-m)$.

- But Step 2. works only if $y_i \sim \mu_x$

Thm 2': Main Idea

- Now can not sample $y_1, \ldots, y_m$ independent of x
- Instead use [HJMR'07] to sample $y_i \sim \mu_x$
  - Each sample costs $I(x;y)$

- Analysis goes through …

4. Contextual Proofs and Uncertainty?

- Scenario: Alice + Bob start with axioms A: subset of clauses on $X_1, \ldots, X_n$
- Alice wishes to prove $A \Rightarrow C$ for some clause C
- But proof $\Pi: A \Rightarrow C$ may be long ($\sim 2^{\sqrt{n}}$)
- Context to rescue: Maybe Alice + Bob share context $D \Leftarrow A$; and contextual proof $\Pi': D \Rightarrow C$ short $(\mathrm{poly}(n))$
- Uncertainty: Alice's Context $D_A \neq D_B$ (Bob's context)
  - Alice writes proof $\Pi': D_A \Rightarrow C$
  - When can Bob verify Π' given D_B?

4. Contextual Proofs and Uncertainty? -2

- Scenario: Alice + Bob start with axioms A: subset of clauses on $X_1, \ldots, X_n$
- Alice wishes to prove $A \Rightarrow C$ for some clause C
  - Alice writes proof $\Pi' : D_A \Rightarrow C$
  - When can Bob verify Π' given D_B?
  - Surely if $D_A \subseteq D_B$
  - What if $D_A \setminus D_B = \{C'\}$ and $\Pi'' : D_B \Rightarrow C'$ is one step long?
  - Can Bob still verify $\Pi' : D_A \Rightarrow C$ in $\mathrm{poly}(n)$ time?
  - Need feasible data structure that allows this!
    - None known to exist. Might be harder than Partial Match Retrieval ...

Summarizing

- Perturbing "common information" assumptions in Shannon/Yao theory, lead to many changes.
  - Some debilitating
  - Some not; but require careful protocol choice.
- In general: Communication protocols are not continuous functions of the common information.
- Uncertain model ($\mathcal{G}$) needs more exploration!
- Some open questions from our work:
  - Tighten the gap: $cc(f) \cdot I$ vs. $cc(f) + \sqrt{I}$
  - Multi-round setting? Two rounds?
  - What if randomness & function imprefectly shared? [Prelim. Results in [Ghazi+S'17]]

Contextual Communication & Uncertainty

Communication Complexity: Yao

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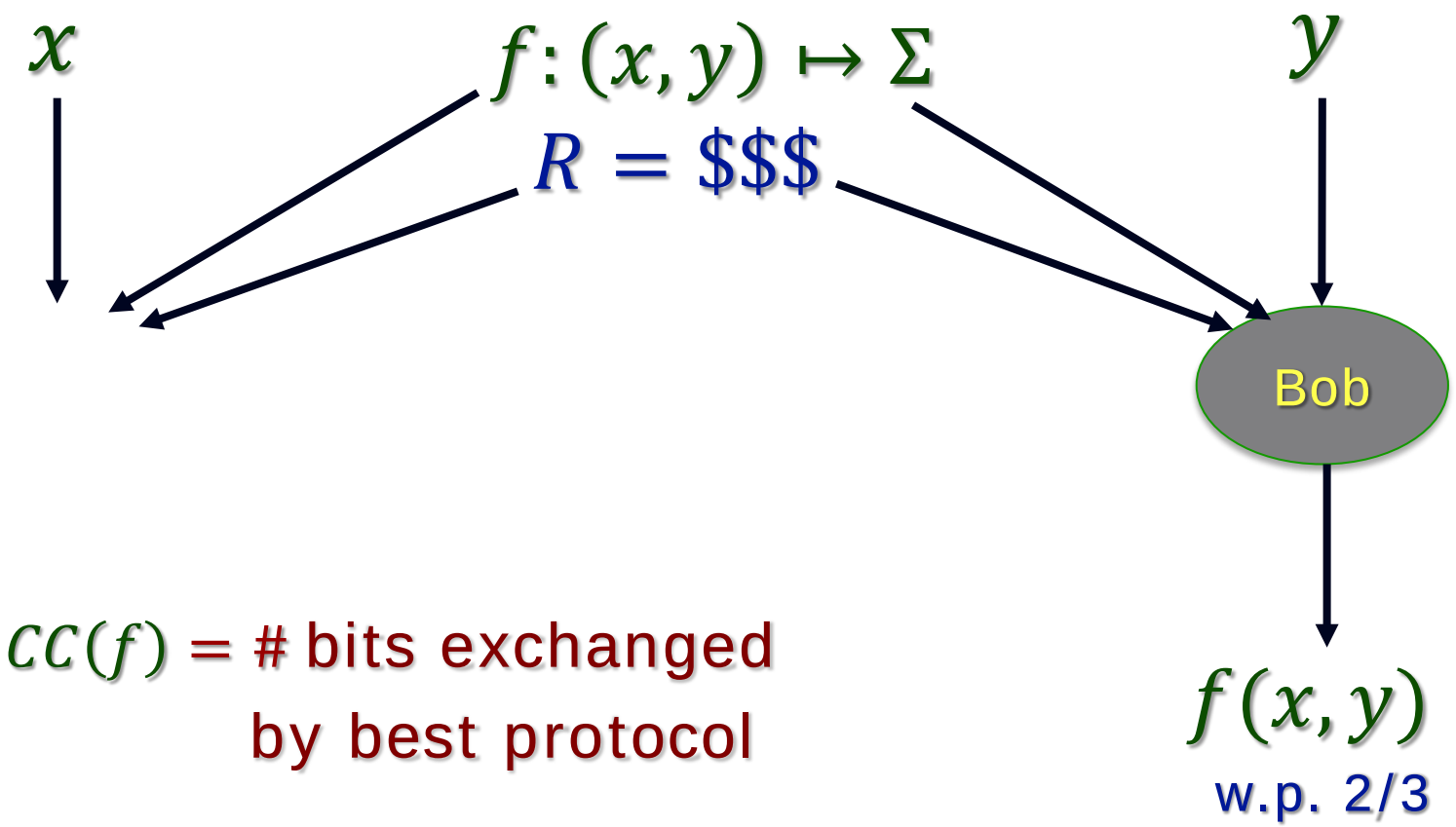

$CC(f)$ = # bits exchanged by best protocol

Uncertain Communication

Boole's "modest" ambition

- *"The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind."* [G.Boole, "On the laws of thought …" p.1]

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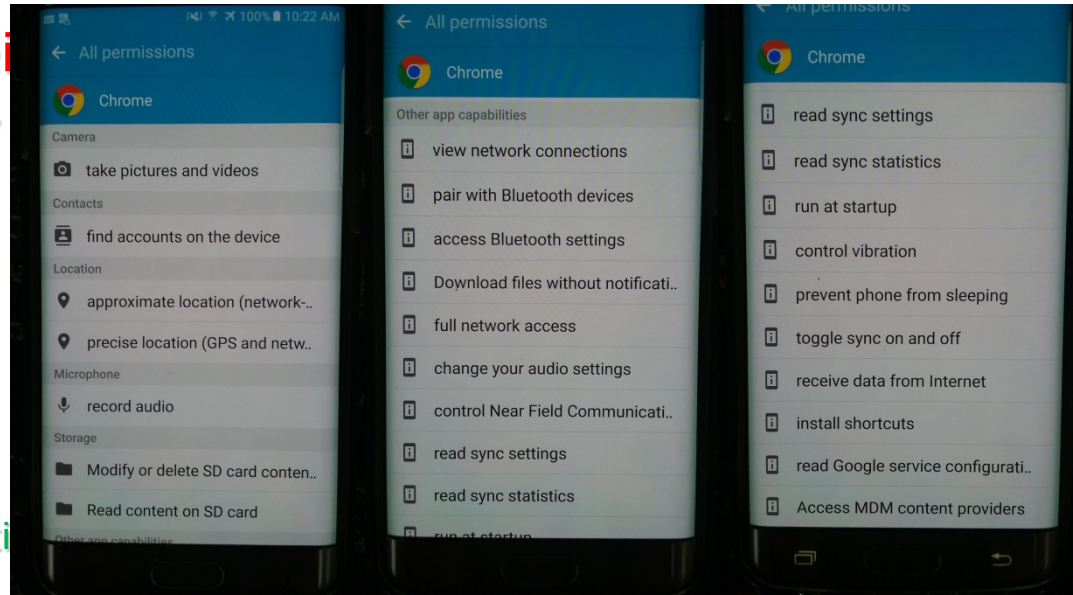
    - Qualitatively (recei
    - and Quantitatively



Aside: Contextual Proofs & Uncertainty

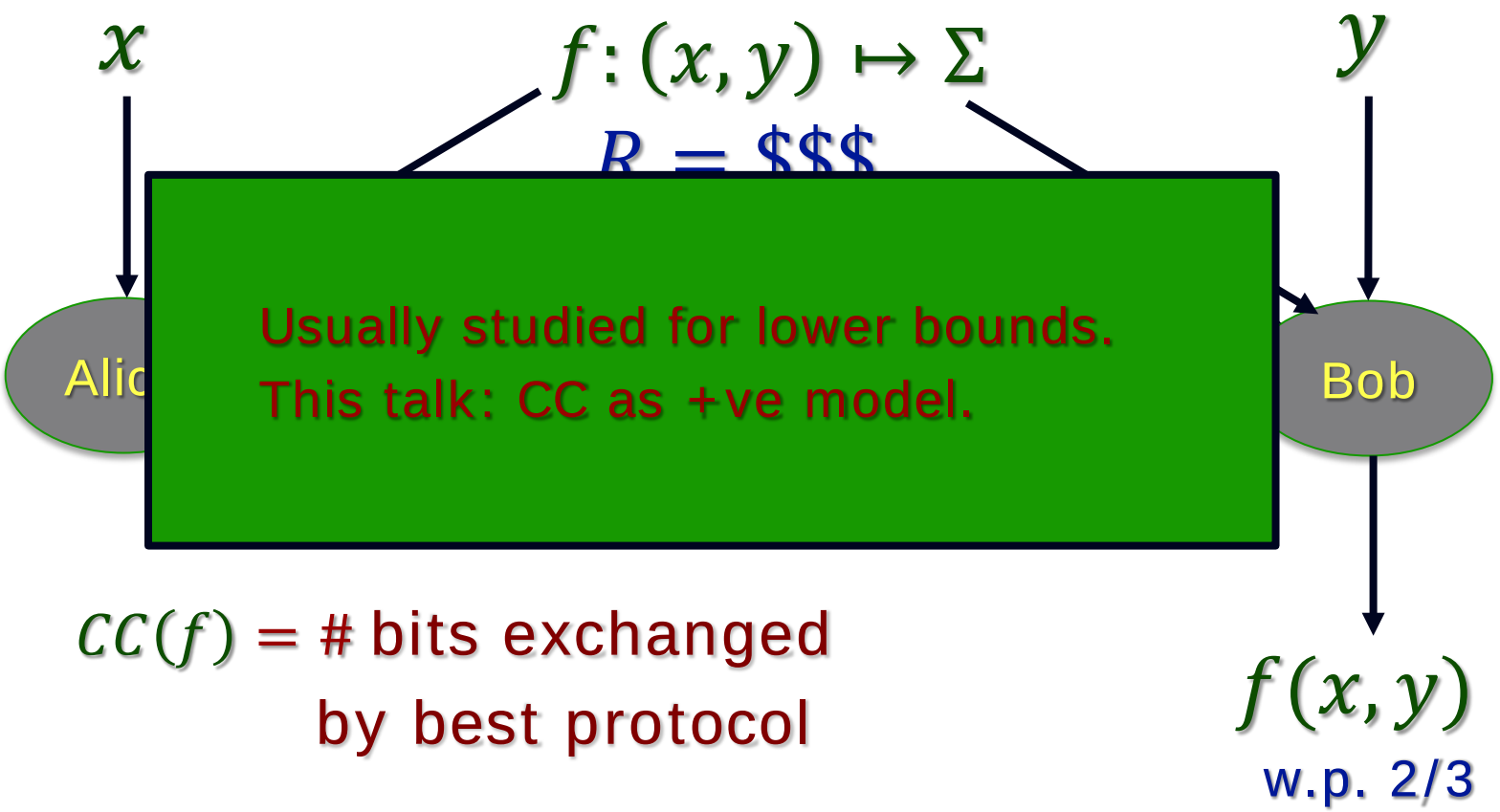
- Mathematical proofs assume large context.
  - "By some estimates a proof that 2+2=4 in ZFC would require about 20000 steps ... so we will use a huge set of axioms to shorten our proofs – namely, everything from high-school mathematics" [Lehman,Leighton,Meyer]
- Context shortens proofs. But context is uncertain!
  - What is "high school mathematics"
    - Is it a fixed set of axioms?
    - Or a set from which others can be derived?
      - Is the latter amenable to efficient reasoning?
      - What is efficiency with large context?

Sales Pitch + Intro

- Most of communication theory [a la Shannon, Hamming]:
  - Built around sender and receiver perfectly synchronized.
  - So large context (codes, protocols, priors) ignored.
- Most Human communication (also device-device)
  - … does not assume perfect synchronization.
  - So context is relevant:
    - Qualitatively (receiver takes wrong action)
    - and Quantitatively (inputs are long!!)
- Theory? What are the problems?
  - Starting point = Shannon? Yao?

Communication Complexity

The model (with shared randomness)



$CC(f)$ = # bits exchanged by best protocol

Aside: Easy CC Problems [Ghazi,Kamath,S'15]

∃ Problems with large inputs and small communication?

- Equality testing:
  - $EQ(x,y) = 1 \Leftrightarrow x = y$;
- Hamming distance:
  - $H_k(x,y) = 1 \Leftrightarrow \Delta(x,y) \leq k$;
- Small set intersection:
  - $\cap_k (x,y) = 1 \Leftrightarrow$ wt(x), wt$(y$
  - $CC(\cap_k) = O(k)$ [Håstad Wi

Protocol:

$poly(k)$ Proto

Use common

to hash $[n] \rightarrow$

$x = (x_1, \dots, x_n)$

$y = (y_1, \dots, y_n)$

$\langle x, y \rangle \triangleq \sum_i x_i y_i$

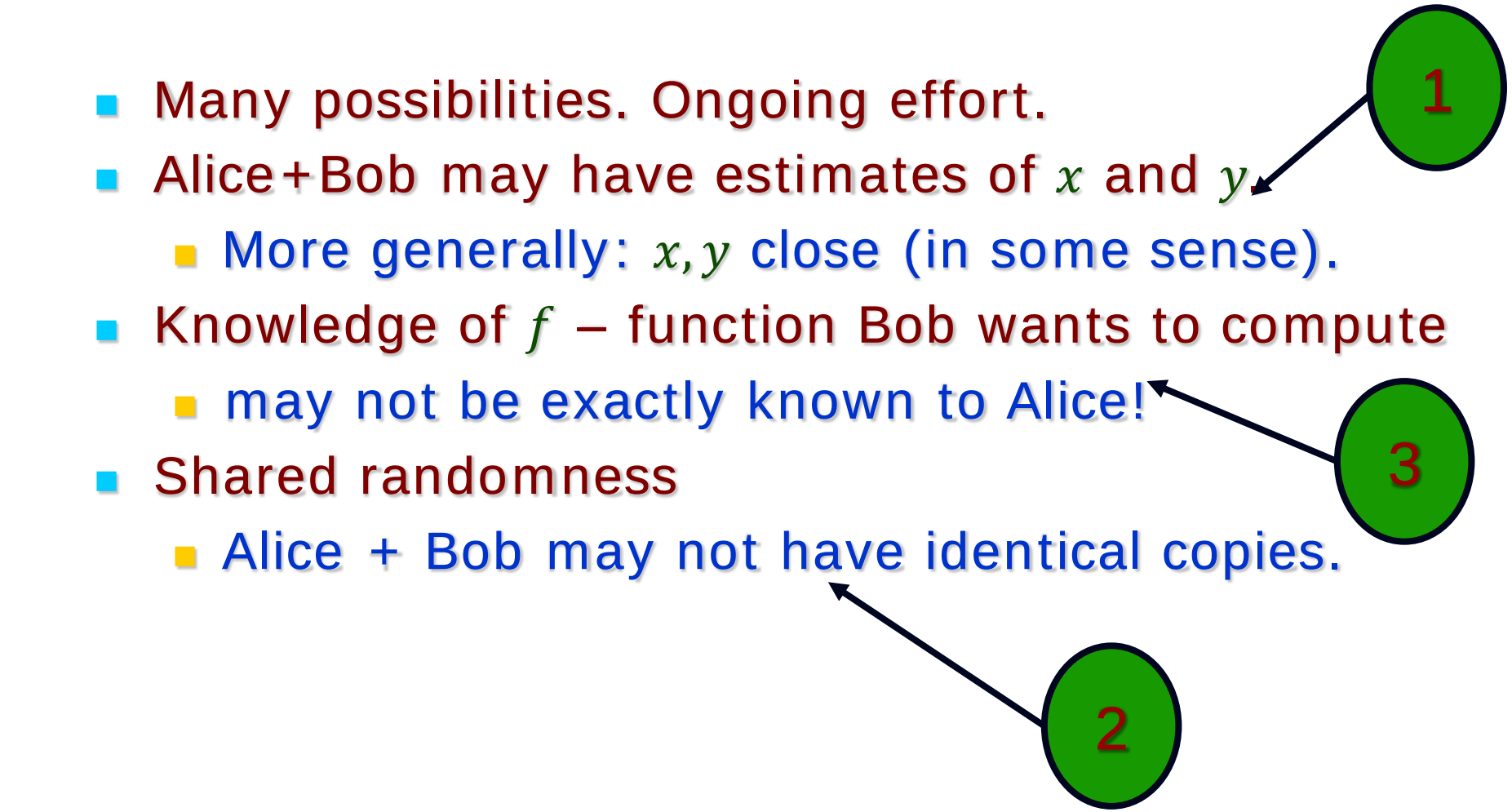
Unstated philosophical contribution of CC a la Yao:

Communication with a focus ("only need to determine $f(x,y)$")

can be more effective (shorter than $|x|, H(x), H(y), I(x;y) \dots$)

Modelling Shared Context + Imperfection

- Many possibilities. Ongoing effort.
- Alice+Bob may have estimates of x and y.
  - More generally: x, y close (in some sense).
- Knowledge of f – function Bob wants to compute
  - may not be exactly known to Alice!
- Shared randomness
  - Alice + Bob may not have identical copies.



1. Compression

- Classical compression: Alice $\leftarrow P, m \sim P$; Bob $\leftarrow P$;
  - Alice $\rightarrow$ Bob: $y = E_P(m)$; Bob $\hat{m} = D_P(y) \stackrel{?}{=} m$;
  - [Shannon]: $\hat{m} = m$; w. $\mathbb{E}_{m \sim P}[|E_P(m)|] \leq H(P) + 1$
    $H(P) \stackrel{\text{def}}{=} \mathbb{E}_{m \sim P}[-\log P(m)]$
- Uncertain compression [Juba,Kalai,Khanna,S.]
  - Alice $\leftarrow P, m \sim P$; Bob $\leftarrow Q$;
  - Alice $\rightarrow$ Bob: $y = E_P(m)$; Bob $\hat{m} = D_Q(y) \stackrel{?}{=} m$;
  - P, Q Δ-close: $\forall m \; |\log P(m) - \log Q(m)| \leq \Delta$
  - Can we get $\mathbb{E}_{m \sim P}[|E_P(m)|] \leq O(H(P) + \Delta)$?
  - [JKKS] – Yes – with shared randomness.
  - [CGMS] – Yes – with shared correlation.
  - [Haramaty+S.] – Deterministically $O(H(P) + \log\log|\Omega|)$

Universe

Deterministic Compression: Challenge

- Say Alice and Bob have rankings of N players.
  - Rankings = bijections $\pi, \sigma : [N] \to [N]$
  - $\pi(i)$ = rank of i^{th} player in Alice's ranking.
- Further suppose they know rankings are close.
  - $\forall\, i \in [N]: |\pi(i) - \sigma(i)| \leq 2$.
- Bob wants to know: Is $\pi^{-1}(1) = \sigma^{-1}(1)$
- How many bits does Alice need to send (non-interactively).
  - With shared randomness – $O(1)$
  - Deterministically?
    - With Elad Haramaty: $\tilde{O}(\log^* n)$

Compression as a proxy for language

- Information theoretic study of language?
- Goal of language: Effective means of expressing information/action.
- Implicit objective of language: Make frequent messages short. Compression!
- Frequency = Known globally? Learned locally?
  - If latter – every one can't possibly agree on it;
  - Yet need to agree on language (mostly)!
  - Similar to problem of Uncertain Compression.
  - Studied formally in [Ghazi,Haramaty,Kamath,S. ITCS 17]

2. Imperfectly Shared Randomness

- Recall: Communication becomes more effective with randomness.
  - Identity, Hamming Distance, Small Set Intersection, Inner Product.
- How does performance degrade if players only share correlated variables:
  - E.g. Alice $\leftarrow r$; Bob $\leftarrow s$. $(r,s) = (r_i,t_i)_i$ i.i.d.

$r_i, s_i \in \{-1,1\}$; $\mathbb{E}[r_i] = \mathbb{E}[s_i] = 0$; $\mathbb{E}[r_i s_i] = \rho \in (0,1)$;

- [CGMS '16]:
  - Comm. With perfect randomness = k

    $\Rightarrow$ Comm. With imperfect randomness = $O_\rho(2^k)$

Imperfectly Shared Randomness

- Easy (Complete) Problem:
  - Gap Inner Product: $x, y \in \mathbb{R}^n$
  - $GIP_{c,s}(x,y) = 1$ if $\langle x, y\rangle \geq \epsilon \cdot |x|_2 \cdot |y|_2$;
    $= 0$ if $\langle x, y\rangle \leq 0$
  - Decidable with $O_\rho\left(\frac{1}{\epsilon^2}\right)$ (o.w.) communication
- Hard Problem:
  - Sparse Gap Inner Product: GIP on sparse x
    - $x \in \{0,1\}^n, y \in \{-1,1\}^n$; $|x|_1 = 2\epsilon n$
  - Classical communication = $O\left(\log\frac{1}{\epsilon}\right)$ [uses sparsity]
  - No way to use sparsity with imperfect randomness.

3. Functional Uncertainty

- [Ghazi,Komargodski,Kothari,S. '16]
- Recall positive message of Yao's model:
  - Communication can be brief, if Alice knows what function $f(x,y)$ Bob wants to compute.
- What if Alice only knows f approximately?
  - Can communication still be short?

The Model

- Recall Distributional Complexity:
  - $(x,y) \sim \mu$; $\text{error}_{\mu}(\Pi) \stackrel{\text{def}}{=} \Pr_{x,y\sim\mu}[f(x,y) \neq \Pi(x,y)]$
  - Complexity: $cc_{\mu,\epsilon}(f) \stackrel{\text{def}}{=} \min_{\Pi: error_{\mu}(\Pi)\leq\epsilon} \left\{\max_{x,y}\{|\Pi(x,y)|\}\right\}$
- Functional Uncertainty Model - I:
  - Adversary picks (f,g). Nature picks $(x,y) \sim \mu$
  - Alice $\leftarrow (f,x)$; Bob $\leftarrow (g,y)$; Compute $g(x,y)$
  - Promise: $\delta_{\mu}(f,g) \stackrel{\text{def}}{=} \Pr_{\mu}[f(x,y) \neq g(x,y)] \leq \delta_0$
  - Goal: Compute (any) $\Pi(x,y)$ with $\delta_{\mu}(g,\Pi) \leq \epsilon_1$
    - (just want $\epsilon_1 \to 0$ as $\delta_0 \to 0$)
- If (f,g) part of input; this is complexity of what?

Modelling Uncertainty

- Modelled by graph $\mathcal{G}$ of possible inputs
- Protocols know $\mathcal{G}$ but not (f, g)
- $cc_{\mu,\epsilon}(\mathcal{G}) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \{cc_{\mu,\epsilon}(g)\}$
- $\delta_\mu(\mathcal{G}) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \{\delta_\mu(f,g)\}$
- Uncertain error:

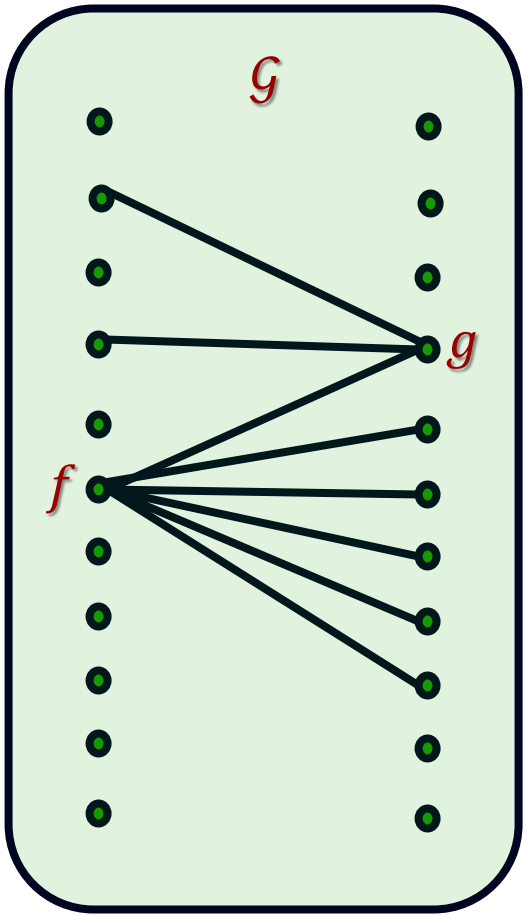
$$error_{\mu,\mathcal{G}}(\Pi) \stackrel{\text{def}}{=} \max_{(f,g)\in\mathcal{G}} \left\{\Pr_{\mu}[g(x,y) \neq \Pi(f,g,x,y)]\right\}$$

- Uncertain complexity:

$$Ucc_{\mu,\epsilon}(\mathcal{G}) \stackrel{\text{def}}{=} \min_{\Pi:\text{error}(\Pi)\leq\epsilon} \left\{\max_{f,g,x,y} \{|\Pi(f,g,x,y)|\}\right\}$$

- Compare $cc_{\mu,\epsilon_0}(\mathcal{G})$ vs. $Ucc_{\mu,\epsilon_1}(\mathcal{G})$

  want $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$



Main Results

- Thm 1: (-ve) $\exists \mathcal{G}, \mu$ s.t. $\delta_\mu(\mathcal{G}) = o(1); cc_{\mu,o(1)}(\mathcal{G}) = 1;$ but $Ucc_{\mu,.1}(\mathcal{G}) = \Omega(\sqrt{n})$; ($n = |x| = |y|$)
- Thm 2: (+ve) $\forall \mathcal{G}$, product μ,

$$Ucc^{oneway}_{\mu,\epsilon_1}(\mathcal{G}) = O\left(cc^{oneway}_{\mu,\epsilon_0}(\mathcal{G})\right)$$

where $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$

Protocols are not continuous wrt the function being computed

- Thm 2': (+ve) $\forall \mathcal{G}, \mu$,

$$Ucc^{oneway}_{\mu,\epsilon_1}(\mathcal{G}) = O\left(cc^{oneway}_{\mu,\epsilon_0}(\mathcal{G}) \cdot (1 + I(x;y))\right)$$

where $\epsilon_1 \to 0$ as $\epsilon_0, \delta_\mu(\mathcal{G}) \to 0$

and $I(x;y)$ = Mutual Information between $x; y$

Details of Negative Result

- μ: $x \sim U(\{0,1\}^n); y = Noisy(x)$; $\Pr[x_i \neq y_i] = 1/\sqrt{n}$;
- $\mathcal{G} = \{(\oplus_S (x \oplus y), \oplus_T (x \oplus y)) |\ |S \oplus T| = o(\sqrt{n})\}$
  - $\oplus_S (z) = \oplus_{i \in S} z_i$
- Certain Comm: Alice → Bob: $\oplus_T (x)$
- $\delta_\mu(\mathcal{G}) = \max_{S,T} \left\{ \Pr_{x,y} [\oplus_S (x \oplus y) \neq \oplus_T (x \oplus y)] \right\}$

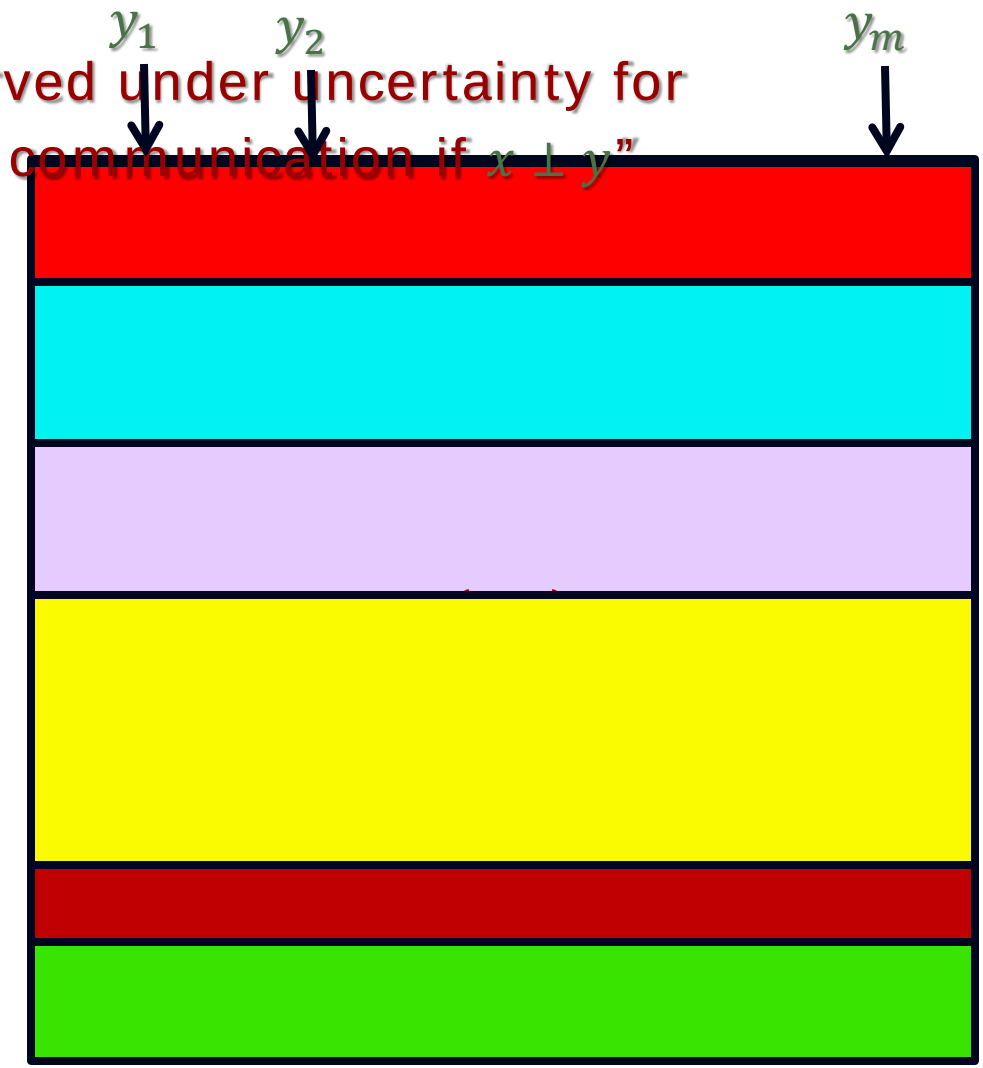
$$= \max_{U:|U|=o(\sqrt{n})} \left\{ \Pr_{z \sim \text{Bernoulli}\left(\frac{1}{\sqrt{n}}\right)^n} [\oplus_U (z) = 1] \right\} = o(1)$$

- Uncertain Lower bound:
  - Standard $cc_{\mu,\epsilon}(F)$ where $F\big((S,x);(T,y)\big) = \oplus_T (x \oplus y)$;
  - Lower bound obtain by picking (S,T) randomly:
    - S uniform; T noisy copy of S

Positive result (Thm. 2)

- Consider comm. Matrix
- Protocol for g partitions matrix in 2^k blocks
- Bob wants to know which block?
- Common randomness: $y_1, \dots, y_m$
- Alice → Bob: $f(x, y_1) \dots f(x, y_m)$
- Bob (whp) recognizes block and uses it.
- $m = O(k)$ suffices.

"CC preserved under uncertainty for one-way communication if $x \perp y$"



Analysis Details

1. W.p. $1-\sqrt{\epsilon}$, $\exists j$ s.t. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \leq \sqrt{\epsilon}$
   - Main idea: If $\Pi_g(x) = \Pi_g(x_j)$ then w.h.p. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \leq \sqrt{\epsilon}$
2. If $j \in [K]$ s.t. $\delta\left(g(x_j,\cdot), g(x,\cdot)\right) \geq 2\sqrt{\epsilon}$ then $\Pr[j \text{ is selected}] = \exp(-m)$.

- But Step 2. works only if $y_i \sim \mu_x$

Thm 2': Main Idea

- Now can not sample $y_1, \ldots, y_m$ independent of x
- Instead use [HJMR'07] to sample $y_i \sim \mu_x$
  - Each sample costs $I(x;y)$

- Analysis goes through …

4. Contextual Proofs and Uncertainty?

- Scenario: Alice + Bob start with axioms A: subset of clauses on $X_1, \ldots, X_n$
- Alice wishes to prove $A \Rightarrow C$ for some clause C
- But proof $\Pi: A \Rightarrow C$ may be long ($\sim 2^{\sqrt{n}}$)
- Context to rescue: Maybe Alice + Bob share context $D \Leftarrow A$; and contextual proof $\Pi': D \Rightarrow C$ short $(\mathrm{poly}(n))$
- Uncertainty: Alice's Context $D_A \neq D_B$ (Bob's context)
  - Alice writes proof $\Pi': D_A \Rightarrow C$
  - When can Bob verify Π' given D_B?

4. Contextual Proofs and Uncertainty? -2

- Scenario: Alice + Bob start with axioms A: subset of clauses on $X_1, \ldots, X_n$
- Alice wishes to prove $A \Rightarrow C$ for some clause C
  - Alice writes proof $\Pi' : D_A \Rightarrow C$
  - When can Bob verify Π' given D_B?
  - Surely if $D_A \subseteq D_B$
  - What if $D_A \setminus D_B = \{C'\}$ and $\Pi'' : D_B \Rightarrow C'$ is one step long?
  - Can Bob still verify $\Pi' : D_A \Rightarrow C$ in $\mathrm{poly}(n)$ time?
  - Need feasible data structure that allows this!
    - None known to exist. Might be harder than Partial Match Retrieval ...

Summarizing

- Perturbing "common information" assumptions in Shannon/Yao theory, lead to many changes.
  - Some debilitating
  - Some not; but require careful protocol choice.
- In general: Communication protocols are not continuous functions of the common information.
- Uncertain model ($\mathcal{G}$) needs more exploration!
- Some open questions from our work:
  - Tighten the gap: $cc(f) \cdot I$ vs. $cc(f) + \sqrt{I}$
  - Multi-round setting? Two rounds?
  - What if randomness & function imprefectly shared? [Prelim. Results in [Ghazi+S'17]]

Thank You!