What should I talk about?

Aspects of Human Communication

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Based on many joint works ...
Ingredients in Human Communication

- Ability to start with (nearly) zero context and “learning to communicate by communicating”.
  - Children learn to speak ... (not by taking courses)
  - Focus of works with Juba and Goldreich
  - “What is possible?” (a la computability)
- Ability to leverage large uncertain context.
  - E.g., ... This talk today ...
  - Assumes ... English, Math, TCS, Social info, Geography.
  - Aside ... what is “self-contained”?
- “How to make communication efficient (using context)?” (a la complexity)
Context in Communication

- **Empirically, Informally:**
  - Huge piece of information (much larger than “content” of communication)
  - Not strictly needed for communication ...
  - ... But makes communication efficient, when shared by communicating players
  - ... helps even if context not shared perfectly.

- **Challenge:** Formalize?
  - Work so far ... some (toy?) settings
Underlying Model: Communication Complexity

The model (with shared randomness)

\[ f : (x, y) \mapsto \Sigma \]

\[ R = $$$ \]

Usually studied for lower bounds. This talk: CC as +ve model.

\[ CC(f) = \# \text{bits exchanged by best protocol} \]

\[ f(x, y) \]

w.p. 2/3
Aside: Easy CC Problems \cite{GKSS15}

\exists \text{ Problems with large inputs and small communication?}

- **Equality testing:**
  \[ EQ(x, y) = 1 \iff x = y; \quad \mathcal{O}(1) \]

- **Hamming distance:**
  \[ H_k(x, y) = 1 \iff \Delta(x, y) \leq k; \quad \mathcal{O}(k) \]

- **Small set intersection:**
  \[ \cap_k (x, y) = 1 \iff \text{wt}(x), \text{wt}(y) \leq k \]
  \[ \mathcal{C}(\cap_k) = O(k) \text{ } \\text{\cite{HW98}} \]

Protocol:

Fix ECC $E \{0, 1\}^n \rightarrow \{0, 1\}^N$.

Use common randomness to hash $[n] \rightarrow poly(k)$.

Accept if $E_x^i = E_y^i$.

\begin{align*}
  x &= (x_1, \ldots, x_n) \\
  y &= (y_1, \ldots, y_n)
\end{align*}

\[ \langle x, y \rangle \triangleq \sum_i x_i y_i \]

Unstated philosophical contribution of CC a la Yao:

Communication with a focus ("only need to determine $f(x, y)$") can be more effective (shorter than $|x|, H(x), H(y), I(x; y)\ldots$)
Communication & (Uncertain) Context

\[ \text{Context}_{\text{Alice}} \equiv (f_A, R_A, P_{XY}) \]
\[ \text{Context}_{\text{Bob}} \equiv (f_B, R_B, P_{XY}) \]

\[ f: (x, y) \mapsto \Sigma \]

\[ f(x, y) \quad \text{w.p.} \ 2/3 \]
1. Imperfectly Shared Randomness

Context$_{Alice} \equiv R^A$

Context$_{Bob} \equiv R^B$

$f: (x, y) \mapsto \Sigma$

$x \xrightarrow{f} \Sigma \xrightarrow{f} y$

$f(x, y)$ w.p. $2/3$
Imperfectly Shared Randomness (ISR)

- Model: $R^A \sim N_\rho(R^B)$ ($\rho$-correlated iid on each coord.)
- Thm [Bavarian-Gavinsky-Ito’15]: Equality testing has $O(1)$-comm. comp. with ISR.
- Thm [Canonne-Guruswami-Meka-S.’16]: If $f$ has CC $k$ with perfect rand., then it has ISR-CC $O_\rho(2^k)$
- Thm [CGMS] This is tight (for promise problems).

- Complete problem: Estimate $\langle x, y \rangle$ for $x, y \in \mathbb{R}^n$
  - $\epsilon$-approximation needs $\Theta(\epsilon^{-2})$ communication
- Hard problem: Sparse inner product – where $x$ is non-zero only $\epsilon$-fraction of the times.
2. Uncertain Functionality

Context_{Alice} \equiv (f, R, P_{X|Y})

Context_{Bob} \equiv (f, R, P_{X|Y})

\( f: (x, y) \mapsto \Sigma \)

\( x \)

\( y \)

Alice

Bob

\( f(x, y) \) w.p. \( \frac{2}{3} \) over \( \frac{2}{3} \sim P_{XY} \)
Definitions and results

- Defining problem is non-trivial:
  - Alice/Bob may not “know” $f$ but protocol might!
  - Prevent this by considering entire class of function pairs $\mathcal{G} = \{(f^A, f^B)\}$ that are admissible.
  - Complexity = complexity of $\mathcal{G}$!

- Theorem[Ghazi,Komargodski,Kothari,S. 16]:
  - If $P_{XY}$ arbitrary then there exists $\mathcal{G}$ s.t. every $f \in \mathcal{G}$ has $cc = 1$, but uncertain-$cc(\mathcal{G}) = \Omega(\sqrt{n})$
  - If $P_{XY} = \text{uniform}$ and every $f \in \mathcal{G}$ has one-way $cc k$, then uncertain-$cc(\mathcal{G}) = O(k)$.

- Theorem[Ghazi-S.,18]: Above needs perfect shared randomness.
3. Compression & (Uncertain) Context

\[ f: (x, y) \mapsto \Sigma \]

\[ \text{Context}_{\text{Alice}} = \frac{P_{X \mid A}}{P_X} \]

\[ \text{Context}_{\text{Bob}} \equiv \frac{P_{X \mid B}}{P_X} \]

\[ f (xx, y) \quad \text{w.p. } 2/3 \]
3. (Uncertain) Compression

- Without context: \( CC = \log |\Omega| \) (where \( x \in \Omega \))
- With shared context, Expected-CC = \( H(P_x) \)
- With imperfectly shared context, but with shared randomness, Expected-CC = \( H(P^A_x) + \Theta(\Delta) \)
  - Where \( \Delta = \max_x \left\{ \max \left\{ \log \frac{P^A(x)}{P^B(x)}, \log \frac{P^B(x)}{P^A(x)} \right\} \right\} \) [JKKS’11]
- Without shared randomness ... exact status unknown! Best upper bound ([Haramaty,S’14]):
  - Expected-CC = \( O(H(P^A_x) + \Delta + \log \log \Omega) \)
Compression as a proxy for language

- Information theoretic study of language?
- Goal of language: Effective means of expressing information/action.
- Implicit objective of language: Make frequent messages short. Compression!
- Frequency = Known globally? Learned locally?
  - If latter – every one can’t possibly agree on it;
  - Yet need to agree on language (mostly)!
  - Similar to problem of Uncertain Compression.
  - Studied formally in [Ghazi, Haramaty, Kamath, S. ITCS 17]
Part II: Proofs
Well understood … (Goldwasser-Micali-Rackoff, …)

- Interactive Proofs
- Zero Knowledge Proofs
- Multi-Prover Interactive Proofs
- PCPs
- Interactive Proofs for Muggles
- Pseudodeterministic Proofs …

… nevertheless some challenges in understanding communication of proofs …
**Standard Assumption**

- **Small (Constant) Number of Axioms**
  - \( X \rightarrow Y, Y \rightarrow Z \Rightarrow X \rightarrow Z, \text{ Peano, etc.} \)

- **Medium Sized Theorem:**
  - \( \forall x, y, z, n \in \mathbb{N}, \quad x^n + y^n = z^n \rightarrow n \leq 2 \ldots \)

- **Big Proof:**
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Mathematical proofs assume large context.

“By some estimates a proof that $2+2=4$ in ZFC would require about 20000 steps ... so we will use a huge set of axioms to shorten our proofs – namely, everything from high-school mathematics”

[Lehman, Leighton, Meyer – Notes for MIT 6.042]

Context (= huge set of axioms) shortens proofs.

But context is uncertain!

What is “high school mathematics”? 

The truth
Communicating (“speaking of”) Proofs

Ingredients:

- **Prover:**
  - Axioms $A$, Theorem $T$, Proof $\Pi$
  - Communicates $(T, \Pi)$ (Claim: $\Pi$ proves $A \rightarrow T$)

- **Verifier:**
  - Complete+sound: $\exists \Pi, V^A(T, \Pi) = 1$ iff $A \rightarrow T$
  - Verifier efficient = $\text{Poly}(T, \Pi)$ with oracle access to $A$

Axioms = Context.
Uncertainty of Context?

- **Prover**: works with axioms $A_P$, generates $\Pi$ s.t. $V^{A_P}(T, \Pi) = 1$
- **Verifier**: works with axioms $A_V$, checks $V^{A_V}(T, \Pi) = 1$
- Robust prover/verifier?
  - Need measure $\delta(A_P, A_V)$ (not symmetric).
  - Given $\delta$ prover should be able to generate $\Pi_\delta$ such that $\forall A_P$ s.t. $\delta(A_P, A_V) \leq \delta$, $V^{A_V}(T, \Pi) = 1$
  - $\Pi_\delta$ not much larger than $\Pi = \Pi_0$
  - $\delta(\ldots)$ “reasonable” ...
    - E.g. if $X \rightarrow Y, Y \rightarrow Z \in A$ and $A' = A \cup \{X \rightarrow Z\}$ then $\delta(A', A)$ tiny.

- Open: Does such an efficient verifier exist?
Beyond oracle access: Modelling the mind

- Brain (of verifier) does not just store and retrieve axioms.
- Can make logical deductions too! But should do so feasibly.
- Semi-formally:
  - Let $A^t$ denote the set of axioms “known” to the verifier given $t$ query-proc. time
  - Want $|A^{2t}| \gg |A^t|$, but storage space $\sim |A^0|$.
- What is a computational model of the brain that allows this?
  - Cell-probe – No*. ConsciousTM – Maybe. Etc...
Conclusions

- Very poor understanding of “communication of proofs” as we practice it.
  - Have to rely on verifier’s “knowledge”
  - But can’t expect to know it exactly
- Exposes holes in computational understanding of knowledge-processing.
  - Can we “verify” more given more time?
  - Or are we just memorizing?
Thank You!