CS 229r Information Theory in Computer Science	Mar 5, 2019
Lecture 11	
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1 Outline

The topic for today's lecture is communication complexity:

1. Upper Bounds
2. Lower Bounds for IP (Inner Product)
   - Distributional Complexity
   - Discrepancy
   - Spectrum

2 Communication Complexity Review

Recall that our model of communication is for Alice and Bob, given $x \in \{0,1\}^n$ and $y \in \{0,1\}^n$ respectively, to send binary strings to each other in rounds in order for Bob to compute $f : \{0,1\}^n \times \{0,1\}^n \to S$ on (x, y), where S is a finite set often chosen to be $\{0,1\}$. We can also add randomness to this model in two ways: either by *public* randomness, where a random string is available to both Alice and Bob simultaneously, or by *private* randomness, where a random string is available to Alice and not Bob and similarly a random string is available to Bob but not Alice.

As before, we have the following definitions.

Definition 1 (Communication Complexity)**.** We define the *communication complexity* of $f : \{0,1\}^n \times \{0,1\}^n \to S$ to be

$$\mathrm{CC}(f) \triangleq \min_{\pi}\{\#\text{ bits exchanged by } \pi\},$$

where the min is taken over all protocols π computing f. Similarly, we define the *private randomness communication complexity* of f to be

$$\mathrm{CC}^{\mathrm{Priv}}(f) \triangleq \min_{\pi}\{\#\text{ bits exchanged by } \pi \text{ with private randomness}\},$$

and the *public randomness communication complexity* of f to be

$$\mathrm{CC}^{\mathrm{Pub}}(f) \triangleq \min_{\pi}\{\#\text{ bits exchanged by } \pi \text{ with public randomness}\}.$$

Note that it's clear from these definitions that

$$\mathrm{CC}^{\mathrm{Pub}}(f) \leq \mathrm{CC}^{\mathrm{Priv}}(f) \leq \mathrm{CC}(f),$$

for any f. We also have the following inequalities in the other direction:

Proposition 2. *For all* $f : \{0,1\}^n \times \{0,1\}^n \to S$*, we have* $\mathrm{CC}^{\mathrm{Priv}}(f) \leq \mathrm{CC}^{\mathrm{Pub}}(f) + O(\log(n))$.

Proposition 3. *For all* $f : \{0,1\}^n \times \{0,1\}^n \to S$*, we have* $\mathrm{CC}(f) \leq 2^{O(\mathrm{CC}^{\mathrm{Priv}}(f))}$.

Note that these two inequalities are tight for $\mathrm{Equality}(x, y)$.

3 Upper Bound Examples

3.1 Hamming Distance

Consider the function

$$\text{HammingDist}_k(x, y) = \begin{cases} 1 & \text{if } \Delta(x, y) \le k, \\ 0 & \text{if } \Delta(x, y) > k, \end{cases}$$

for some parameter k, where $\Delta(x, y) = \#\{i : x_i \neq y_i\}$ is the Hamming distance between $x, y \in \{0, 1\}^n$. It turns out that there is a $\Theta(k \log k + 1)$ bit protocol with shared randomness (does not depend on n). Today, we will see a $\Theta(k^2 + 1)$ bit protocol with shared randomness. Note that if $k = 0$, this is the Equality function, which we know has $\Theta(1)$ public randomness communication complexity, so this protocol is reasonably tight for small k.

3.2 Small Set Disjointness

Consider the Small Set Disjointness problem, where Alice gets $S \subseteq [n]$ and Bob gets $T \subseteq [n]$ (both represented as characteristic vectors) with the condition that $|S|, |T| \le k$ for some parameter k. The goal is to output whether $S \cap T = \emptyset$. Hastad and Wigderson give a $\Theta(k)$ bit protocol, but we will see a $\Theta(k \log k)$ bit protocol today.

3.3 Protocols using hash functions

Both of these problems can be solved by protocols that publicly pick a completely random hash function $h : [n] \to [m]$, which can be shown to have the property that for all $W \subseteq [n]$ with $|W| \le k$, we have

$$\Pr_h[\exists i \neq j \in W \text{ s.t. } h(i) = h(j)] \le \frac{1}{100}.$$

for some $m = O(k^2)$.

Exercise 4. *Prove that a unformly random function $h : [n] \to [m]$ satisfies the above property for some $m = O(k^2)$.*

For Small Set Disjointness, we can apply this to $W = S \cup T$, and Alice can send $\{h(i)\}_{i \in S}$ to Bob, which takes $|S| \log m \le O(k \log k)$ bits. Since the probability of any collision is small, we know that Bob can recover S with high enough probability and thus compute whether $S \cap T = \emptyset$.

For HammingDist_k, for all $j \in [m]$, Alice can compute

$$u_j = \bigoplus_{i \in h^{-1}(j)} x_i$$

and send the message $\{u_j\}_{j \in [m]}$. Then, Bob can similarly compute

$$v_j = \bigoplus_{i \in h^{-1}(j)} y_i,$$

and check whether $\Delta(u, v) \le k$. If $\Delta(x, y) \le k$, then x and y differ in at most k indices $\subseteq \{i_1, \ldots, i_k\}$, which implies that u and v differ only on a subset of the indices $\{h(i_1), \ldots, h(i_k)\}$, which implies $\Delta(u, v) \le k$. If $\Delta(x, y) > k$, then one can show that $\Delta(u, v) > k$ with probability $\ge 2/3$, which completes the analysis of this $\Theta(k^2 + 1)$ bit protocol for HammingDist_k.

3.4 "Distance" problems in $\mathbb{R}^n$

Here, Alice and Bob are given $x, y \in \mathbb{R}^n$ respectively with $\|x\|_2 = \|y\|_2 = 1$. First, consider the function

$$f(x,y) = \sum_{i=1}^{n} x_i - y_i$$

where we allow an additive error of up to ε.

Remark The requirement that $\|x\|_2 = \|y\|_2 = 1$ is only so that the error term ε makes sense, as otherwise, we could scale x and y up without any change in ε, which would be too good to be true.

For this function, the protocol is easy: Alice sends $(\sum x_i) \pm \varepsilon$ in $O(\log(1/\varepsilon))$ bits, and Bob can compute the rest.

What about the function

$$f(x,y) = \sum_{i=1}^{n} (x_i - y_i)^2$$

with an additive error of up to ε? Here, the cross-terms $x_i y_i$ cause us difficulty. However, with randomness, Alice and Bob can overcome this obstacle. Specifically, Alice can send $(\sum x_i^2) \pm \varepsilon$, similar to before, and now she can also send $\sum R_i x_i$, where $R_1, \ldots, R_n$ are "bits" identically and independently distributed uniformly over $\{-1, 1\}$. For Bob to decode this, note that

$$\begin{aligned} \mathop{\mathbb{E}}_R \left[\left(\sum_i R_i x_i \right) \left(\sum_j R_j y_j \right) \right] &= \mathop{\mathbb{E}}_R \left[\sum_i R_i^2 x_i y_i \right] + \mathop{\mathbb{E}}_R \left[\sum_{i \neq j} R_i R_j x_i y_j \right] \\ &= \sum_i x_i y_i + 0, \end{aligned}$$

where the last equality comes from the fact that $R_i^2 = 1$ and $\mathbb{E}_R[R_i R_j] = 0$ for all $i \neq j$. Therefore, Bob can take $(\sum_i R_i x_i)$ from Alice and $\left(\sum_j R_j y_j\right)$ directly from its input and multiply them to get an estimate for $\sum_i x_i y_i$. Given that Alice sends $\sum_i x_i^2$ and Bob can deduce $\sum_j y_j^2$, Bob can estimate $\sum_j (x_j - y_j)^2$. For this to work with high probability, we need to squash the variance of the random variable $\sum_{i \neq j} R_i R_j x_i y_j$. We can squash this variance successfully with $O(1/\varepsilon^2)$ bits of communication. In fact, this is the best we can hope for:

Exercise 5. *Prove that $1/\varepsilon^2$ bits are required for any protocol to compute $f(x,y) = \sum_{i=1}^{n} (x_i - y_i)^2$ up to an additive error of ε.*

In summary, for the function $f(x,y) = (\sum_{i=1}^{n} x_i - y_i) \pm \varepsilon$, there is a protocol that uses $O(\log(1/\varepsilon))$ bits, but for the function $f(x,y) = (\sum_{i=1}^{n} (x_i - y_i)^2) \pm \varepsilon$, the best protocol uses $\Theta(1/\varepsilon^2)$ bits.

4 Lower Bounds for Inner Product

Recall that IP is defined as

$$\mathrm{IP}(x,y) \triangleq \sum_{i=1}^{n} x_i y_i \mod 2,$$

for $x, y \in \{0,1\}^n$. How can we prove a $\Omega(n)$ lower bound on communication complexity of IP with shared randomness? One avenue to pursue would be to look at the rank of the matrix M_{IP}, but we saw for Equality that rank was not helpful in proving lower bounds for protocols with randomness.[1] So, we need something new.

4.1 Distributional Complexity

Idea: Put a distribution μ on $\{0,1\}^n \times \{0,1\}^n$. We can define

$$\delta_\mu(f,g) \triangleq \Pr_{(x,y)\sim\mu}[f(x,y) \neq g(x,y)]$$

and

$$D_{\varepsilon,\mu}(f) \triangleq \min_{g \text{ s.t. } \delta_\mu(f,g)\leq\varepsilon} \mathrm{CC}(g).$$

4.2 Randomized Protocol $\implies$ Distributional Deterministic Protocol

With this setup, we can prove distributional lower bounds by putting some distribution μ on $\{0,1\}^n \times \{0,1\}^n$, and prove that no deterministic protocol π using k bits achieves small error on $(x,y)\sim\mu$.

Why is this helpful?

Proposition 6. *For all functions $f : \{0,1\}^n \times \{0,1\}^n \to S$ and distributions μ over $\{0,1\}^n \times \{0,1\}^n$, we have*

$$\mathrm{CC}^{\mathrm{Pub}}(f) \geq \frac{D_{\varepsilon,\mu}(f)}{O(\log(1/\varepsilon))}.$$

Thus, if we have a lower bound on k for any *deterministic* protocol computing f achieving small error for some distribution μ, then we must have a lower bound for any *random* protocol with public randomness computing f.

Proof of Proposition 6. Suppose we have some k-bit protocol π that gets error less than $1/3$ probability for every $(x,y) \in \{0,1\}^n \times \{0,1\}^n$. By repeating this protocol $O(\log(1/\varepsilon))$ times and taking the majority of the outputs, we have a protocol $\tilde{\pi}$ using $O(k\log(1/\varepsilon))$ bits that errors with probability $\leq \varepsilon$. That is, for all (x,y), we have

$$\mathop{\mathbb{E}}_R[\mathbb{1}_{f(x,y)\neq\tilde{\pi}(x,y,R)}] \leq \varepsilon,$$

where the randomness R denotes the randomness of the protocol. Now, we can take the expectation over μ and switch the order to get

$$\begin{aligned}\varepsilon &\geq \mathop{\mathbb{E}}_{(x,y)\in\mu}\mathop{\mathbb{E}}_R[\mathbb{1}_{f(x,y)\neq\tilde{\pi}(x,y,R)}] \\ &= \mathop{\mathbb{E}}_R\mathop{\mathbb{E}}_{(x,y)\in\mu}[\mathbb{1}_{f(x,y)\neq\tilde{\pi}(x,y,R)}].\end{aligned}$$

This means that there exists some R such that $\mathbb{E}_{(x,y)\in\mu}[\mathbb{1}_{f(x,y)\neq\tilde{\pi}(x,y,R)}] \leq \varepsilon$, i.e. $\Pr_{(x,y)\in\mu}[f(x,y) \neq \tilde{\pi}(x,y,R)] \leq \varepsilon$. Now, we can hardcode R into $\tilde{\pi}$ to get a *deterministic* protocol π' using $O(k\log 1/\varepsilon)$ bits, where we have $\Pr_{(x,y)\in\mu}[f(x,y) \neq \pi'(x,y)] \leq \varepsilon$, i.e. $\delta_\mu(f,\pi') \leq \varepsilon$, as desired. $\square$

The idea here is that we can view randomized protocols as distributions over deterministic protocols.

[1] There is a related quantity to rank called *approximate rank*, whose log lower bounds randomized communication complexity. However, it was shown in 2018 [1] that the log of approximate rank and randomized communication complexity are not polynomially related, refuting the log-approximate-rank conjecture.

4.3 Discrepancy

Now, we would like to show $D_{\mu,\varepsilon}(\mathrm{IP}_n) \geq \Omega(n)$ for some distribution μ, as from the proposition above, this would give a $\Omega(n)/\log(1/\varepsilon)$ lower bound on the number of bits of any protocol computing IP_n with public randomness. In this case, thankfully choosing μ to be uniform will suffice, i.e. $\mu(x,y) = 4^{-n}$ for all $x, y \in \{0,1\}^n$.

Suppose π is a protocol for f using k bits, with error probability $\leq \varepsilon$ over μ (or equivalently, $D_{\mu,\varepsilon}(f) \leq k$). Without loss of generality, we can assume that the final bit communicated by π is the function value (as this adds at most 1 round and 1 bit). Considering the usual matrix M_{IP}, we know that the k bit protocol splits the matrix into $K = 2^k$ rectangles $R_1, \ldots, R_K$, where by a rectangle, we mean a Cartesian product of some $S \subseteq [n]$ and $T \subseteq [n]$. Let p_i denote the probability that π is correct and ends up in rectangle R_i, and let ε_i denote the probability that π is wrong and ends up in rectangle R_i. Then, we have

$$\sum_{i=1}^{K} p_i \geq 1 - \varepsilon,$$
$$\sum_{i=1}^{K} \varepsilon_i \leq \varepsilon.$$

Subtracting the two inequalities, we have $\sum_{i=1}^{K} p_i - \varepsilon_i \geq 1 - 2\varepsilon$, which implies that for some $i \in [K]$, we have

$$p_i - \varepsilon_i \geq \frac{1-2\varepsilon}{K} = \frac{1-2\varepsilon}{2^k}. \tag{1}$$

Now, we are ready for another definition. In addition to the matrix $M_f(x,y) = f(x,y) \in \{0,1\}$ as we saw in the last lecture, we can now define

$$M_{f,\mu}(x,y) \triangleq \mu(x,y)(-1)^{f(x,y)}.$$

Translating equation (1) into this new notation, for rectangle R_i, which we can say is given by rectangle $S \times T$, we have

$$\left| \sum_{x,y \in \{0,1\}^n} \mathbb{1}_S(x)\mathbb{1}_T(y) M_{f,\mu}(x,y) \right| = |p_i - \varepsilon_i| \geq \frac{1-2\varepsilon}{2^k}.$$

This motivates the following definition:

Definition 7 (Discrepancy)**.** We can define the *discrepancy* of f with respect to μ to be

$$\mathrm{Disc}_\mu(f) \triangleq \max_{S,T \subseteq [n]} \left| \sum_{x,y} \mathbb{1}_S(x)\mathbb{1}_T(y) M_{f,\mu}(x,y) \right|.$$

We have just shown:

Proposition 8. *If $D_{\mu,\varepsilon}(f) \leq k$, then we have*

$$\mathrm{Disc}_\mu(f) \geq \frac{1-2\varepsilon}{2^k}.$$

Our goal now is to show that $\mathrm{Disc}_\mu(\mathrm{IP}_n)$ is small, as this would imply $D_{\mu,\varepsilon}(f)$ is big by (the contrapositive of) proposition 8, which would imply that $\mathrm{CC}^{\mathrm{Pub}}(f)$ is big by Proposition 6.

4.4 Spectrum bounds Discrepancy

We can bound $\mathrm{Disc}_\mu(\mathrm{IP}_n)$ directly, where we represent S, T by characteristic column vectors $U, V \in \{0,1\}^{2^n}$. Recall that μ is uniform over $\{0,1\}^n \times \{0,1\}^n$. We have

$$\begin{aligned}
\mathrm{Disc}_\mu(\mathrm{IP}_n) &= \max_{S,T\subseteq[n]} \left| \sum_{x,y} \mathbb{1}_S(x)\mathbb{1}_T(y) M_{\mathrm{IP}_n,\mu}(x,y) \right| && (2)\\
&= \max_{U,V\in\{0,1\}^{2^n}} \left| U^\top M_{\mathrm{IP}_n,\mu} V \right| && (3)\\
&\leq \max_{\substack{U,V\in\mathbb{R}^{2^n}\\ \|U\|_2,\|V\|_2\leq 2^{n/2}}} \left| U^\top M_{\mathrm{IP}_n,\mu} V \right| && (4)\\
&= 2^n \max_{\substack{U,V\in\mathbb{R}^{2^n}\\ \|U\|_2,\|V\|_2\leq 1}} \left| U^\top M_{\mathrm{IP}_n,\mu} V \right| && (5)\\
&= 2^n \lambda_{\max}(M_{\mathrm{IP}_n,\mu}). && (6)
\end{aligned}$$

Thankfully, $M_{\mathrm{IP}_n,\mu}$ has enough structure to make computing its maximum eigenvalue tractable. In fact,

Exercise 9. $M_{\mathrm{IP}_n,\mu_n} = (M_{\mathrm{IP}_1,\mu_1})^{\otimes n}$, *where* μ_i *is uniform over* $\{0,1\}^i \times \{0,1\}^i$.

Corollary 10. $\lambda_{\max}(M_{\mathrm{IP}_n,\mu_n}) = (\lambda_{\max}(M_{\mathrm{IP}_1,\mu_1}))^n$.

We can explicitly write M_{IP_1,μ_1} as

$$M_{\mathrm{IP}_1,\mu_1} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix}$$

as $\mu_1 = 1/4$ for all inputs, and $(-1)^{xy}$ is -1 if $x = y = 1$ and 1 otherwise. A computation shows that $\lambda_{\max}(M_{\mathrm{IP}_1,\mu_1}) = 1/\sqrt{8}$, so $\lambda_{\max}(M_{\mathrm{IP}_n,\mu_n}) = (1/\sqrt{8})^n$. Thus, plugging back into (6), we get

$$\mathrm{Disc}_\mu(\mathrm{IP}_n) \leq 2^n \lambda_{\max}(M_{\mathrm{IP}_n,\mu}) = 2^{-n/2}.$$

Thus, for $k = n/2 - 1$ and $\varepsilon < 1/4$, we can apply the contrapositive of Proposition 8 to get that $D_{\mu,\varepsilon}(\mathrm{IP}_n) \geq n/2 - 1$. For constant $\varepsilon < 1/4$ and applying Proposition 6, we have $\mathrm{CC}^{\mathrm{Pub}}(\mathrm{IP}_n) \geq \Omega(n)$, as desired.

References

[1] Chattopadhyay, Arkadev, Nikhil S. Mande, and Suhail Sherif. The log-approximate-rank conjecture is false. *Electronic Colloquium on Computational Complexity (ECCC)*, 2018.