

Limits of Local Algorithms in Random Graphs

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Main Result

- Background: Almost surely, random d -regular graph on n vertices has independent set of size $(1 + o(1)) \cdot c_d \cdot n$ for $c_d = \frac{2}{d} \log d$.
- Can you find such a large independent set?
 - Greedy finds one of half this size.
- Our Theorem: “Local algorithms” can not. In fact they fall short by a constant factor.

Definition: Local Algorithms

- Informally: Local algorithms
 - Input = Communication network.
 - Wish to use local communication to compute some property of input.
 - In our case – large independent set in graph.
 - Allowed to use randomness, generated locally.

Formally

- (Randomized) Decision Algorithm:
 - $f(u, G, \vec{w}) \in \{0,1\}$: Determines if $u \in I$.
 - \vec{w} is a weighting, say in $[0,1]$, on vertices
- Correctness:
 - $\forall u, v, G, \vec{w}$ s.t. $u \leftrightarrow_G v$,
 $f(u, G, w) = 0$ or $f(v, G, w) = 0$.
- Locality:
 - f is r -local if $f(u, G, \vec{w}) = f(v, H, \vec{x})$ whenever r -local weighted neighborhood around u in (G, \vec{w}) and v in (H, \vec{x}) are identical.

Locality \neq Locality

- LOCALity in distributed algorithms
 - Usually algorithms try to compute some function of input graph, on the graph itself.
 - Algorithm uses data available topologically locally.
 - Leads to our model
- Locality a la Codes/Property Testing
 - Locality simply refers to number of queries to input.
 - More general model.
 - We can't/don't deal with it.

Motivations for our work

1. Paucity of “complexity” results for random graphs.
Major exceptions:
 - Rossman: AC^0 /Monotone complexity of planted clique.
 - Feige-Krauthgamer/SOS relaxations.
2. Physicists explanation of complexity
 - Many heuristically successful algorithms only have “local” rigorous explanations.
 - Clustering/Shattering explain inability of algorithms.
3. Remarkable success of Local algorithms [Onak,Nguyen '08]
4. Graph Limit theory
 - Local characteristics of (random) graphs predict global properties (nearly).

Motivations (contd.)

- Specific conjecture [Hatami-Lovasz-Szegedy]:
As $r \rightarrow \infty$, r -local algorithms should find independent sets of cardinality $c_d(1 - o(1))n$.
- Refuted by our theorem.

Proof

- Part I:
 - A clustering phenomenon for independent sets in random graphs [Inspired by Coja-Oghlan].
- Part II:
 - $\text{Locality} \Rightarrow \text{Continuity} \Rightarrow \neg(\text{Clustering})$.

Both parts simple.

Clustering Phenomena

- Generally:
 - When you look at “near-optimal” solutions, then they are very structured.
 - \Rightarrow topology of solutions highly disconnected (in Hamming space).
- In our context
 - Consider graph on independent sets (of size $\approx c_d n$) with $I \leftrightarrow J$ if $|I \Delta J| \leq \epsilon \cdot n$.
 - Highly disconnected?

Clustering Theorem

- Theorem: $\forall d, \exists 0 < \theta < \tau < c_d$ s.t.:
 - Almost surely over G , $\forall I, J$ of size $\approx c_d n$,

$$\frac{|I \cap J|}{n} \notin (\theta, \tau)$$

- Proof:
 - Compute expected number of independent sets with forbidden intersection and note it is $\ll 1$.
 - Second moment proves concentration.
- Implies Clustering.

Locality $\Rightarrow \neg(\text{Clustering})$

- Main Idea:
 - Fix r -local function f , that usually produces independent sets of size $\approx c_d \cdot n$
 - Sample weights twice: \vec{w} , and then \vec{x} ; p -correlatedly.
 - Let $I = f(G, \vec{w})$ and $J = f(G, \vec{x})$.
 - Prove:
 - whp, $|I|, |J| \approx c_d \cdot n$
 - whp, $|I \cap J| \approx \beta(p) \cdot n$
 - $\exists p$ s.t. $\beta(p) \in (\theta, \tau)$

Size of Ind. Set

- Claim: Size of independent set produced by local algorithms is concentrated.
 - Let $\alpha = \alpha(f) = \mathbb{E}_{\vec{w}}[f(u, \mathbb{T}_d, \vec{w})]$
(where \mathbb{T}_d = infinite tree of degree d)
 - W.p. $1-o(1)$, size of ind. set produced $\approx \alpha \cdot n$.
- Proof:
 - Most neighborhoods are trees \Rightarrow Expectation.
 - Most neighborhoods are disjoint \Rightarrow Chebychev.

p -correlated distributions

- Pick $\vec{w}, \vec{y} \in [0,1]^n$, independently.
- Let $\vec{x}_i = \vec{w}_i$ w.p. p and \vec{y}_i otherwise, independently for each i .
- Let $\beta(p) = \mathbb{E}_{\vec{w}, \vec{x}} [f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$
- As in previous argument:
 - $\mathbb{E}[|I \cap J|] \approx \beta(p) \cdot n$
 - $|I \cap J|$ concentrated around expectation.

Continuity of $\beta(p)$

- Fix \vec{w}, \vec{y} , and consider

$$\Pr[f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$$

- Above expression is some polynomial in p , of degree at most d^r .
- In particular, it is continuous as function of p .
- $\Rightarrow \beta(p)$ = Expectation over \vec{w}, \vec{y} is also continuous.
- Suffices to show $[\beta(0), \beta(1)] \cap (\theta, \tau) \neq \emptyset$.

Continuity (contd.)

- $\beta(p) = \mathbb{E}_{\vec{w}, \vec{x}} [f(u, \mathbb{T}_d, \vec{w}) \wedge f(u, \mathbb{T}_d, \vec{x})]$
- $\beta(1) = \alpha(f) \approx c_d$
- $\beta(0) = \alpha^2 \approx c_d^2$
- Follows from calculations (also naturally) that
 - $[\beta(0), \beta(1)] \cap (\theta, \tau) \neq \emptyset$
- Conclude:
 - whp, $|I|, |J| \approx c_d \cdot n$
 - whp, $|I \cap J| \approx \beta(p) \cdot n$
 - $\exists p$ s.t. $\beta(p) \in (\theta, \tau)$

Some extensions

- Our Theorem: Only rules out $\gamma \cdot OPT$ for some $\gamma < 1$. Don't rule out every $\gamma > 1/2$.
- Settled by [Rahman-Virag'17]
 - “list-decoding” ingredient to analysis.
 - Extension to “sequential decimation” algorithms for NAE K-SAT [Gamarnik,S. '17]
- Extensions to MaxCut in 4-unif. hypergraphs [Chen,Gamarnik,Panchenko,Rahman'19]
- More to come ... [Gamarnik, Jagannath]

Conclusions

- “Clustering” is an obstacle?
- Answer:
 - At least to local algorithms.
 - Local algorithms behave continuously, forcing non-clustering of solutions.
- Open questions:
 - Barrier to local algorithms in general sense?
 - To other complexity classes?

Thank You