Streaming & Sketching CSPs

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Based on joint works with Chi-Ning Chou, Alexander Golovnev, Noah Singer, Ameya Velingker and Santhoshini Velusamy.
This Talk

- CSPs (& approximation & streaming/sketching)
- Our results
- Some Proof Ideas
Constraint Satisfaction Problems (CSPs)

- Class of infinitely many problems.

- Specified by \( q, k \) and family \( F \subseteq \{f:[q]^k \rightarrow \{0,1\}\} \).

Instance of MaxCSP\((F)\): \( \Psi = (X_1, \ldots, X_n; C_1, \ldots, C_m) \); Constraint \( C_j(X) = f_j(X_{i_1(j)}, \ldots, X_{i_k(j)}); f_j \in F \)

\[ \text{opt}(\Psi) \equiv \max_{a \in [q]^n} \left\{ \sum_j C_j(a) \right\} \]

- Examples:
  - MaxCut \( (F = \{\oplus\}) \)
  - MaxDicut \( (F = \{x \land \overline{y}\}) \)
  - Max 3SAT \( (|F| = 11) \), Max Exact 3SAT \( (|F| = 4) \), Max \( q \)-Colorability.
Constraint Satisfaction Problems (CSPs)

“Special Case”: Boolean MaxCSP($F$):

- $q = 2$;
- Constraints $C = f(X_1, \overline{X_2}, \ldots, \overline{X_k})$
  - Constraints applied to literals.

Warning: MaxCSP($F$) ≠ Boolean MaxCSP($F$)

$\forall F \exists G$ s.t. Boolean MaxCSP($F$) = MaxCSP($G$).
(converse not true. MaxCut, MaxDicut ...)

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Streaming & Approximation

- **Streaming input**
  - Streaming: $s(n)$-space algorithm - gets one constraint at a time.
  - Sketching: Maintains sketch $S(\sigma)$ with $|S(\sigma)| \leq s(n)$;
  - Restriction: $(S(\sigma), S(\tau)) \rightarrow S(\sigma \circ \tau)$
  - Space milestones: polylog, sqrt, or (nearly-)linear.

- **Approximations**:
  - Usual notion: $\alpha$-approximation:
    - Output $v$ s.t. $\alpha \cdot \text{opt}(\Psi) \leq v \leq \text{opt}(\Psi)$
  - Refined notion: $(\gamma, \beta)$-distinguishability:
    - Output Yes if $\text{opt}(\Psi) \geq \gamma$, No if $\text{opt}(\Psi) \leq \beta$
  - Equivalence: $\alpha = \min_{\gamma} \max_{\beta} \beta / \gamma$
Trivial Approximations:

- $\tilde{O}(n)$-space, $1 - o(1)$-approx.
  - Maintain $\tilde{O}(n)$-constraints. Solve optimally on those using exponential time.

- $O(1)$-space, $\rho_{\text{min}}$-approx.
  - Defn: $\rho_{\text{min}}(F) := \min \{\text{val}_\psi\}$
  - Notes: Usually $\rho_{\text{min}} > 0$ (unless $0 \in F$)
  - For Boolean CSPs (on literals)
    - $\rho_{\text{min}}(f) = 2^{-k} \cdot |f^{-1}(1)|$ ( = value of random assgmt).
    - E.g. $\rho_{\text{min}}(\text{MaxCut}) = \frac{1}{2}$
  - Key question: (When) can we do better than trivial?
Why study CSPs

- Contain some problems of direct interest
  - Max Cut, Max Dicut, Max colorability
- Allow possibility of classification!
- Highlight general algorithms
  - Norm approximations (already used)
  - Local Exploration
  - Crude snapshots
- Identify Phenomena:
  - No $2^{\sqrt{\log n}}$-space algorithms?
Brief History

- 2011 Bertinoro W’shop (P. 45): “We know nothing +/-”
- 2015-19: Lower bounds for MaxCut [KKS, KKSV, KK]
  - Kapralov-Krachun: $\frac{1}{2} + \epsilon$-approximation requires $\Omega(n)$-space (in $n$ vertex graph) (streaming).
- 2017-20: Algorithms for Max DiCut, Max SAT
  - Guruswami-Velingker-Velusamy: DiCut
  - Chou-Golovnev-Velusamy: DiCut, Max SAT
- 2020: Sketching Classification
  - Chou-Golovnev-Velusamy: Classify all Boolean MaxCSP with $k = q = 2$
Our Results

- **[CGSV21]:**
  - Dichotomy for sketching (polylog vs. sqrt)
  - Polylog space algorithms for infinitely many CSPs
  - $\Omega(\sqrt{n})$ space lower bounds for broad classes ("one-wise ind.", "padded one-wise ind.").

- **[CGSVV 21]:** Linear space lower bounds for subclass of "one-wise-ind".
  - Pins approximability of all MaxCSP($F$) to within q-factor (trivial alg vs. linear space).

- **[SSV21]:** No sublinear algorithms for "Ordering CSPs"
Proof Ideas
Streaming MaxCut
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Max Cut Lower Bound

- **Hard distributions:**
  - YES: Random union of matchings crossing hidden bipartition
  - NO: Random union of matchings.

- **Analysis:**
  - Divide long stream into $O(1)$ smaller substreams – each substream = matching.
  - Algorithm learns nothing in any single substream $\iff$ Boolean Hidden Matching Lower Bound [GKKRW]
  - Hybrid argument to combine $O(1)$ substreams.
  - Yields $\Omega(\sqrt{n})$ space lower bound to beat trivial approximation.
  - $\Omega(n)$ lower bound more complex – omitted.
Boolean Hidden Matching Problem

- One-way communication problem.
- Alice gets a random cut on vertex set $[n]$.
- Bob gets a random matching on $[n]$ of size $\alpha n$ along with a 0/1 label on each edge.
  - NO: 0/1 labels random
  - YES: 1 ⇒ edge crosses cut, 0 ⇒ doesn’t cross.
- Challenge: Alice sends message to Bob, Bob to distinguish YES from NO.
- Lower bound theorem [GKKRW]: $\Omega(\sqrt{n})$ communication required.
GVV+CGV Algorithms for Max DiCut

- Define $\text{Bias}(v) \overset{\text{def}}{=} \text{indeg}(v) - \text{outdeg}(v)$
- $\text{Bias}(G) \overset{\text{def}}{=} \frac{1}{2} \sum_v |\text{Bias}(v)|$
- Claim 1 [GVV]:
  - Bias can be estimated in polylog space. ($\ell_1$-norm estimation)
  - $\text{Dicut}(G) \leq \frac{\text{Bias}(G) + m}{2}$
  - $\text{Bias}(G) \leq \text{Dicut}(G)$ (Greedy rounding)
  - Output: $\max \left\{ \frac{m}{4}, \text{Bias}(G) \right\} \Rightarrow 2/5$-approx.

- Claim 2 [CGV]: $\text{Bias}(G) \leq \frac{m}{3} \Rightarrow \text{Dicut}(G) \geq \frac{m}{4} + \frac{\text{Bias}(G)^2}{4(m-2\text{Bias}(G))}$
  (Rand. Rounding w.p. $\frac{1}{2} - \frac{\text{Bias}(G)}{2(m-2\text{Bias}(G))}$) $\Rightarrow 4/9$-approx.
Generalizing to other CSPs: Challenges

- What is bias, say for $f(x, y, z) = x \land (y \oplus z)$
  - E.g.: $x \land (a_1 \oplus a_2), a_3 \land (x \oplus y), y \land (x \oplus a_4)$

- Why is $\ell_1$-estimation useful in Dicut approximation
  - $\ell_1(x_1, ..., x_n) = \max_{a_1...a_n \in \{-1,1\}} \{\sum_i a_i x_i\}$
  - Useful generalization: $\|M\|_{1,\infty} := \max_{a \in [q]^n} \{\sum_i M_{i,a(i)}\}$
  - Computable in polylog space

- Dicut Analysis: Graph theory, some 3-var. calculus, Why did rounding end up optimal?
Dichotomy for Sketching

- Stepping back: Suppose algorithm gets entire "incidence matrix" $M$ (and only this info)

  ($M_{ij} =$ fraction of constraints with $X_i$ in $j$th place in constraint.)

- How well can this algorithm perform?
Dichotomy for Sketching

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  \[
  (M_{ij} = \text{fraction of constraints with } X_i \text{ in } j\text{th place in constraint})
  \]

- How well can this algorithm perform?

- Tautology: \( \exists \Psi_1, \Psi_2 \text{ with } \text{val}(\Psi_1) \geq \gamma, \text{val}(\Psi_2) \leq \beta \)
  
  and \( M(\Psi_1) = M(\Psi_2) \iff \text{algorithm can’t solve } (\gamma, \beta)\)-MAX CSP(f).
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  $M_{ij} =$ fraction of constraints with $X_i$ in $j$th place in constraint.

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- Thm: - Alg can be sketched in polylog space;
  - If Alg can’t solve then no $o(\sqrt{n})$-sketching alg
Dichotomy for Sketching

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- Thm: - Alg can be sketched in polylog space;
  
  - If Alg can’t solve then no $o(\sqrt{n})$-sketching alg
  
  - Criterion is decidable in finite time.
Decidability & Criterion

- Main insight: Suffices to look at instances on $kq$ variables.
- Instance = distribution of constraints over $kq$ variables.
- ...
- $\exists$ convex sets $K_Y^Y$ and $K_N^N \subseteq \Delta([q])^k \subseteq \mathbb{R}^{qk}$ s.t. Algorithm exists if $K_Y^Y \cap K_N^N = \emptyset$
Complexity?

- If Algorithm does not exist, then?
  - There exist distributions $D^Y$ and $D^N \in \Delta([q]^k)$ with matching (one-wise) marginals corresponding to two “interesting” instances.

- Theorem: If such $D^Y$ and $D^N$ exist then $o(\sqrt{n})$ space sketching algorithm can’t solve $(\gamma, \beta)$-distinguishability!

- One-wise Theorem: If such $D^Y$ and $D^N$ exist with $D^N = \text{Unif}([q]^k)$ then $o(\sqrt{n})$ space streaming algorithm can’t solve $(\gamma, \beta)$-distinguishability!

Extends to Padded One Wise ($k=q=2$):

$$D^Y = \tau D_0 + (1-\tau)D_1; D^N = \tau D_0 + (1-\tau)\text{Unif}([q]^k)$$
Underlying Communication Problem

- Randomized Mask Detection:
  - Alice $\leftarrow X \sim \text{Unif}([q]^n)$
  - Bob $\leftarrow$ Seq. of disjoint masked projections of $X$

  - Projection: $X \rightarrow (S, X|_S)$ $S \subseteq [n]$, $|S| = k$
  - Masked: $(S, X|_S) \rightarrow (S, X|_S + m \mod q)$
    - YES: $m \sim D^Y$; NO: $m \sim D^N$
  - Sequence: $S_1, \ldots, S_{0001n}$
  - Disjoint: $S_i \cap S_j = \emptyset$
  - Challenge: Distinguish YES from NO

- Thm: Requires $\Omega(\sqrt{n})$ bits Alice $\rightarrow$ Bob if marginals of $D^Y$ and $D^N$ match.

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Proof ingredients

- **Step 1:** If $D_N = Unif([q]^k)$ then $(D_Y, D_N)$-RMD hard.
  - Proof extends Fourier analytic proof of GKKRW.

- **Step 1.5:** $(Unif(\{u, v\}), Unif(\{\max(u, v), \min(u, v)\}))$-RMD hard

- **Step 2:** General $(D_Y, D_N)$:
  - $(D, D - \epsilon Unif(\{u, v\}) + \epsilon Unif(\{\max(u, v), \min(u, v)\}))$-RMD hard
  - Can go from $D_Y$ to $D_N$ in $N(q, k)$ steps.
Open Questions

- Sketching = Streaming?
  - Extend sqrt lower bounds from sketching to streaming!
  - Challenge: Walk length algorithm!
- Sqrt = Linear?
  - Is there a dichotomy at linear space?
  - Challenge: Template-based algorithms!
  - Can Dicut approximation be improved with o(n) space?
  - \( \omega(n) \)-lower bounds? (Trivial: \( O(n \log n) \) ...)

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Thank You!