Streaming & Sketching CSPs

Madhu Sudan
Harvard University

Based on joint works with Chi-Ning Chou, Alexander Golovnev, Noah Singer, Ameya Velingker and Santhoshini Velusamy.

This Talk

- CSPs (& approximation & streaming/sketching)
- Our results
- Some Proof Ideas

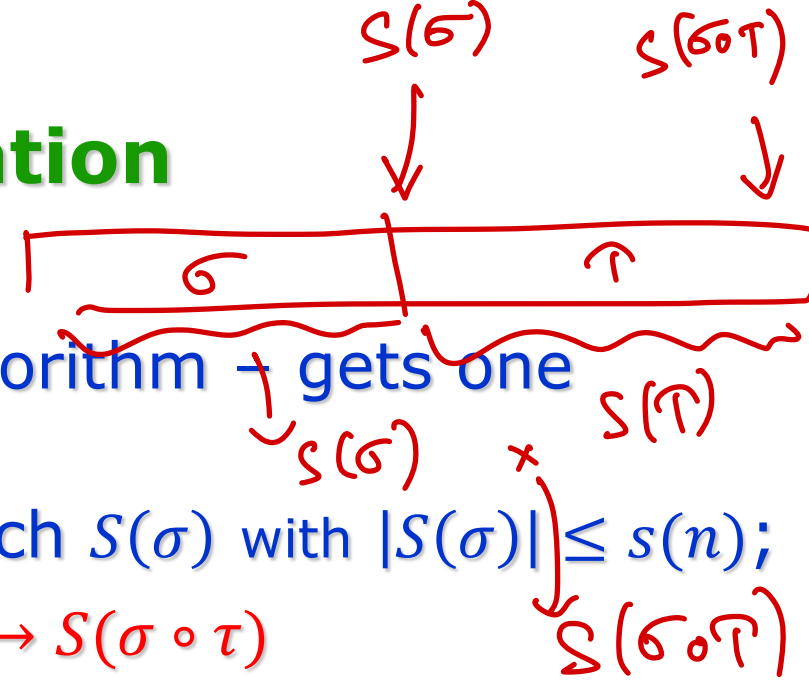
Constraint Satisfaction Problems (CSPs)

- Class of infinitely many problems. $\boxed{\text{MaxCSP}(F)}$
- Specified by q, k and family $F \subseteq \{f : [q]^k \to \{0,1\}\}$.
- Instance of MaxCSP(F): $\Psi = (X_1, \dots, X_n; C_1, \dots, C_m)$;
  Constraint $C_j(X) = f_j(X_{i_1(j)}, \dots, X_{i_k(j)})$; $f_j \in F$
  - $opt(\Psi) \equiv \max_{a \in [q]^n} \{\sum_j C_j(a)\}$, $\frac{1}{m}$
- Examples:
  - MaxCut ($F = \{\oplus\}$)
  - MaxDicut ($F = \{x \wedge \bar{y}\}$)
  - Max 3SAT ($|F| = 11$), Max Exact 3SAT ($|F| = 4$), Max q-Colorability.

Constraint Satisfaction Problems (CSPs)

- "Special Case": Boolean MaxCSP(F):
  - $q = 2$;
  - Constraints $C = f(X_1, \overline{X_2}, \ldots, \overline{X_k})$
    - Constraints applied to <u>literals</u>.
  - Warning: MaxCSP(F) $\neq$ Boolean MaxCSP(F)
  - $\forall F\ \exists G$ s.t. Boolean MaxCSP(F) = MaxCSP(G). (converse not true. MaxCut, MaxDicut …)

Streaming & Approximation



- Streaming input
  - Streaming: $s(n)$-space algorithm + gets one constraint at a time.
  - Sketching: Maintains sketch $S(\sigma)$ with $|S(\sigma)| \leq s(n)$;
    - Restriction: $(S(\sigma), S(\tau)) \to S(\sigma \circ \tau)$
  - Space milestones: polylog, sqrt, or (nearly-)linear.
- Approximations:
  - Usual notion: α-approximation:
    - Output v s.t. $\alpha \cdot opt(\Psi) \leq v \leq opt(\Psi)$
  - Refined notion: (γ, β)-distinguishability:
    - Output Yes if $opt(\Psi) \geq \gamma$, No if $opt(\Psi) \leq \beta$
  - Equivalence: $\alpha = \min_{\gamma} \max_{\beta} \beta / \gamma$

Trivial Approximations:

- $\tilde{O}(n)$-space, $1 - o(1)$-approx.
  - Maintain $\tilde{O}(n)$-constraints. Solve optimally on those using exponential time.
- $O(1)$-space, $\rho_{\min}$-approx.
  - Defn: $\rho_{\min}(F) \coloneqq \min_{\Psi} \{\mathrm{val}_{\Psi}\}$
  - Notes: Usually $\rho_{\min} > 0$ (unless $0 \in F$)
  - For Boolean CSPs (on literals)
    - $\rho_{\min}(f) = 2^{-k} \cdot |f^{-1}(1)|$ (= value of random assgmt).
    - E.g. $\rho_{\min}(\mathrm{MaxCut}) = \frac{1}{2}$
- Key question: (When) can we do better than trivial?

Why study CSPs

- Contain some problems of direct interest
  - Max Cut, Max Dicut, Max colorability
- Allow possibility of classification!
- Highlight general algorithms
  - Norm approximations (already used)
  - Local Exploration
  - Crude snapshots
- Identify Phenomena:
  - No $2^{\sqrt{\log n}}$-space algorithms?

Brief History

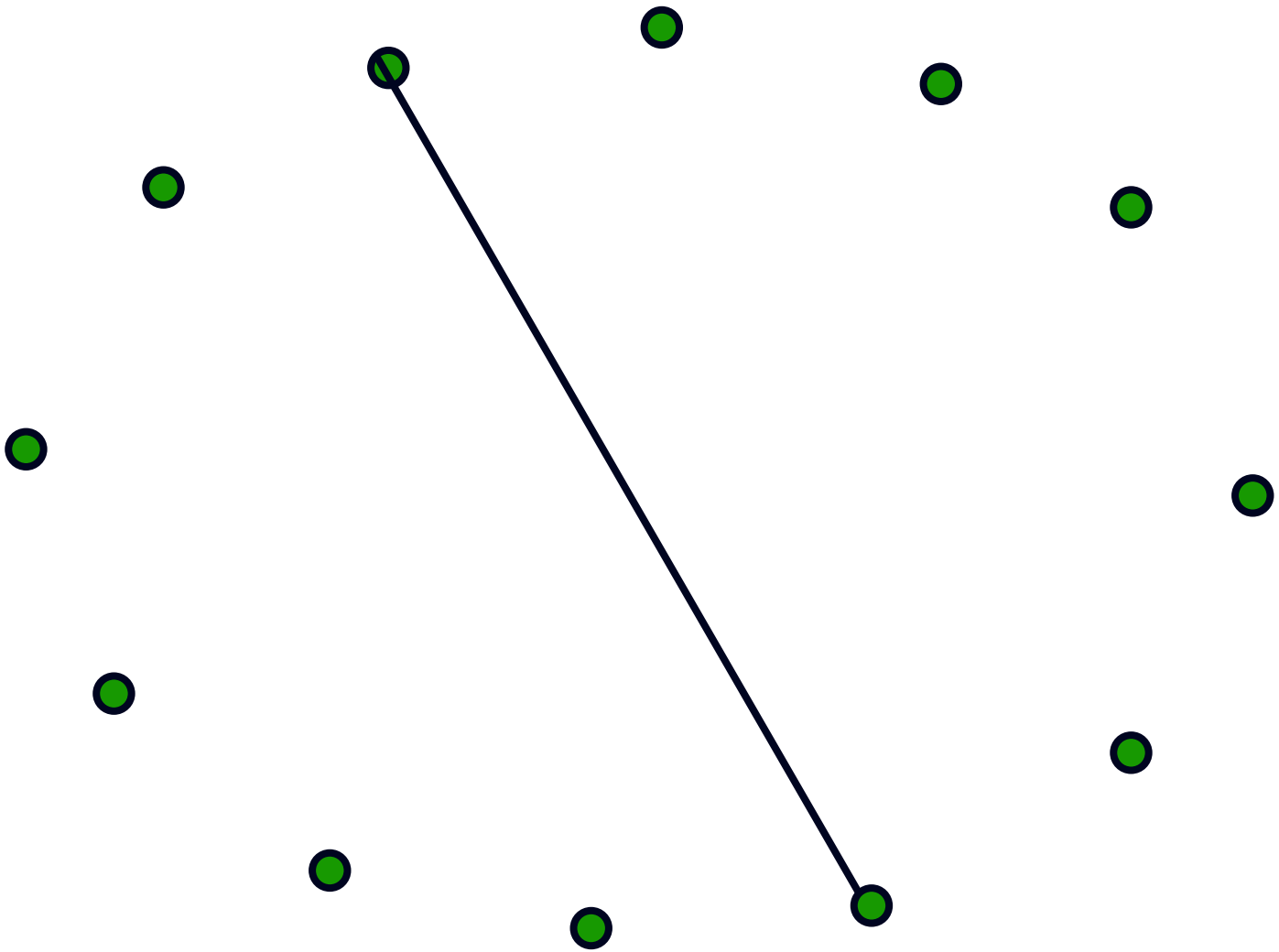
- 2011 Bertinoro W'shop (P. 45): "We know nothing +/-"
- 2015-19: Lower bounds for MaxCut [KKS,KKSV,KK]
  - Kapralov-Krachun: $\frac{1}{2} + \epsilon$-approximation requires $\Omega(n)$-space (in n vertex graph) (streaming).
- 2017-20: Algorithms for Max DiCut, Max SAT
  - Guruswami-Velingker-Velusamy: DiCut
  - Chou-Golovnev-Velusamy: DiCut, Max SAT
- 2020: Sketching Classification
  - Chou-Golovnev-Velusamy: Classify all Boolean MaxCSP with $k = q = 2$

Our Results

- [CGSV21]:
  - Dichotomy for sketching (polylog vs. sqrt)
  - Polylog space algorithms for infinitely many CSPs
  - $\Omega(\sqrt{n})$ space lower bounds for broad classes ("one-wise ind.", "padded one-wise ind.").
- [CGSVV 21]: Linear space lower bounds for subclass of "one-wise-ind".
  - Pins approximability of all MaxCSP(F) to within q-factor (trivial alg vs. linear space).
- [SSV21]: No sublinear algorithms for "Ordering CSPs"

Proof Ideas

Streaming MaxCut



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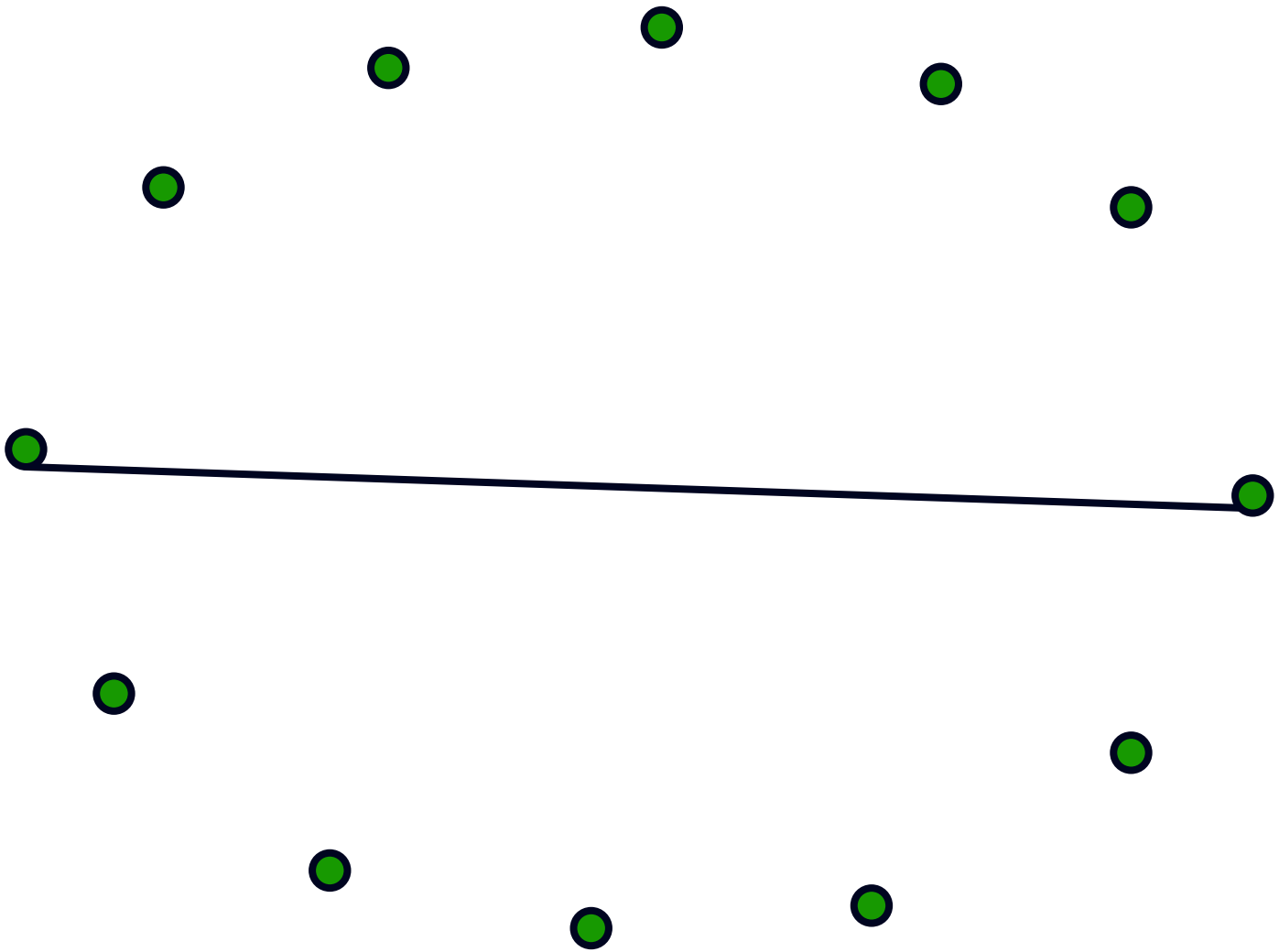

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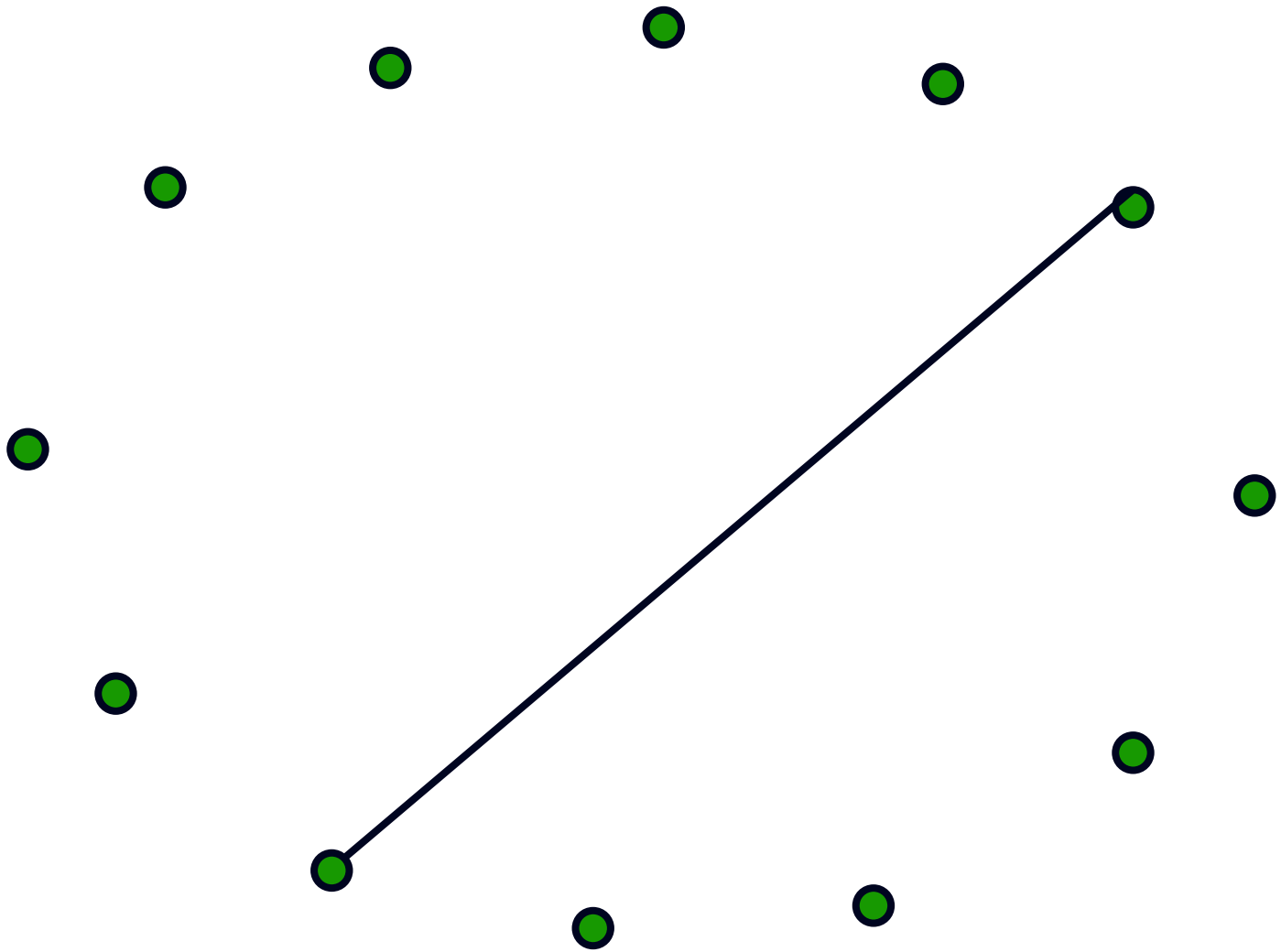

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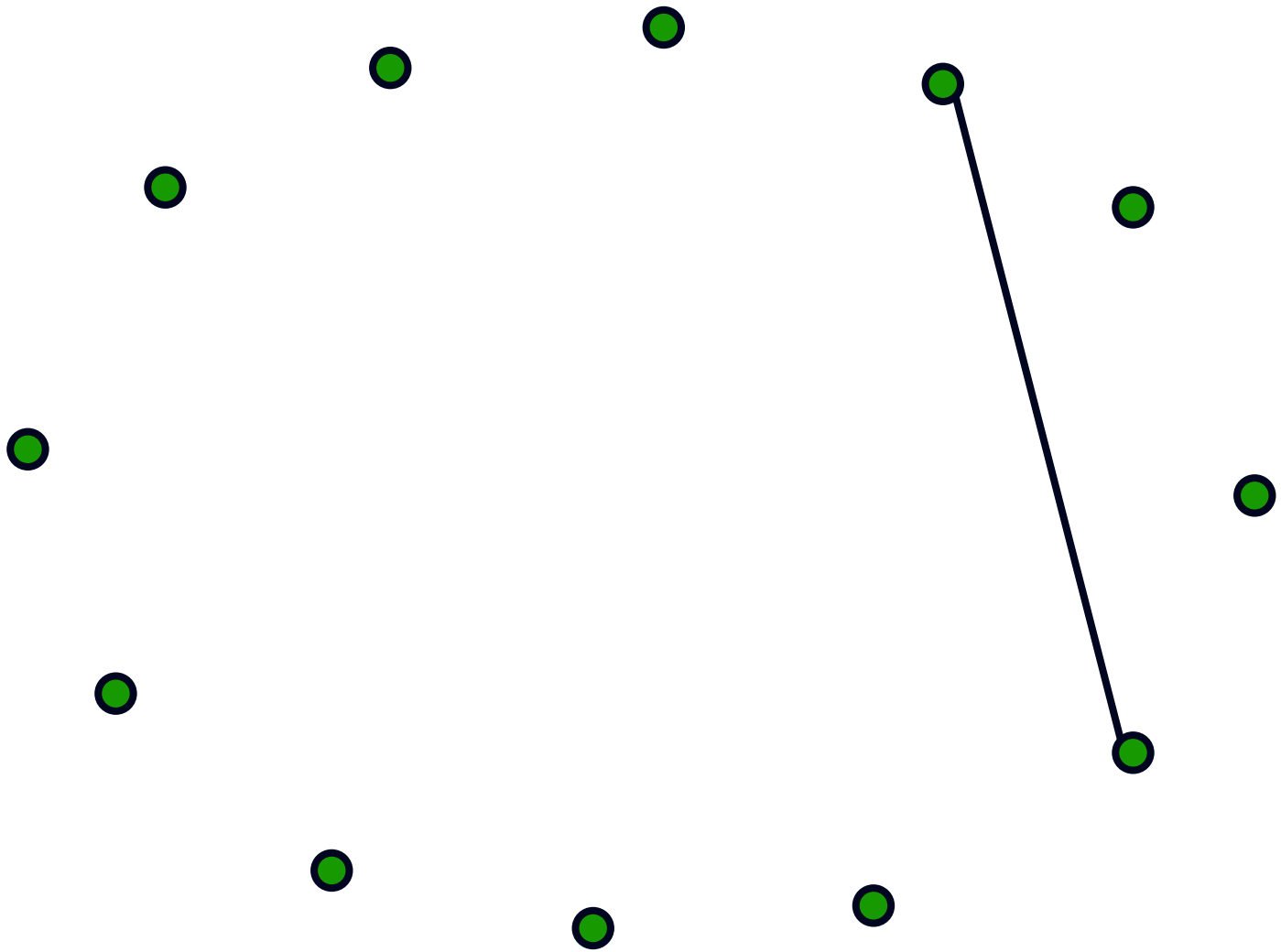

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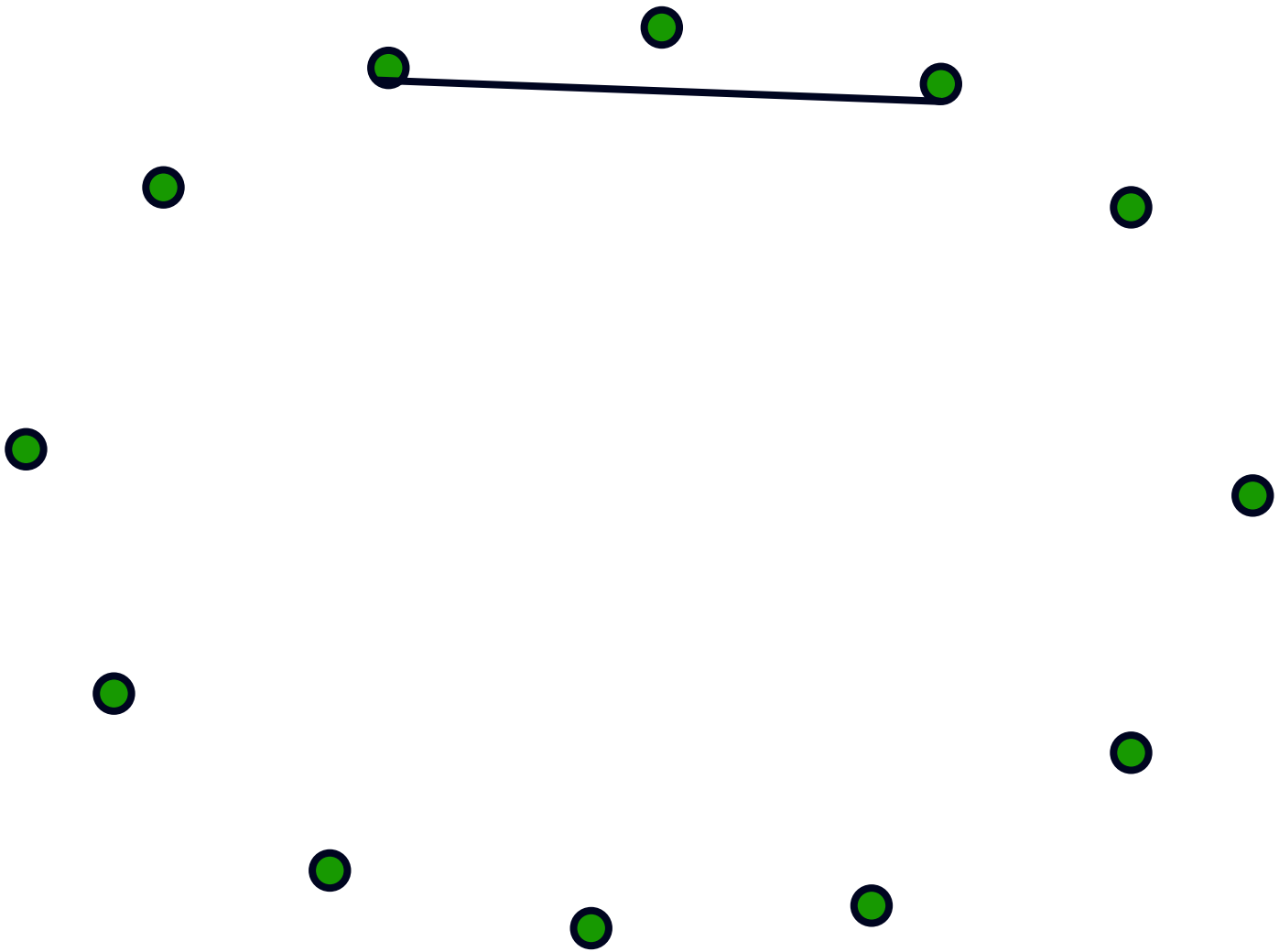

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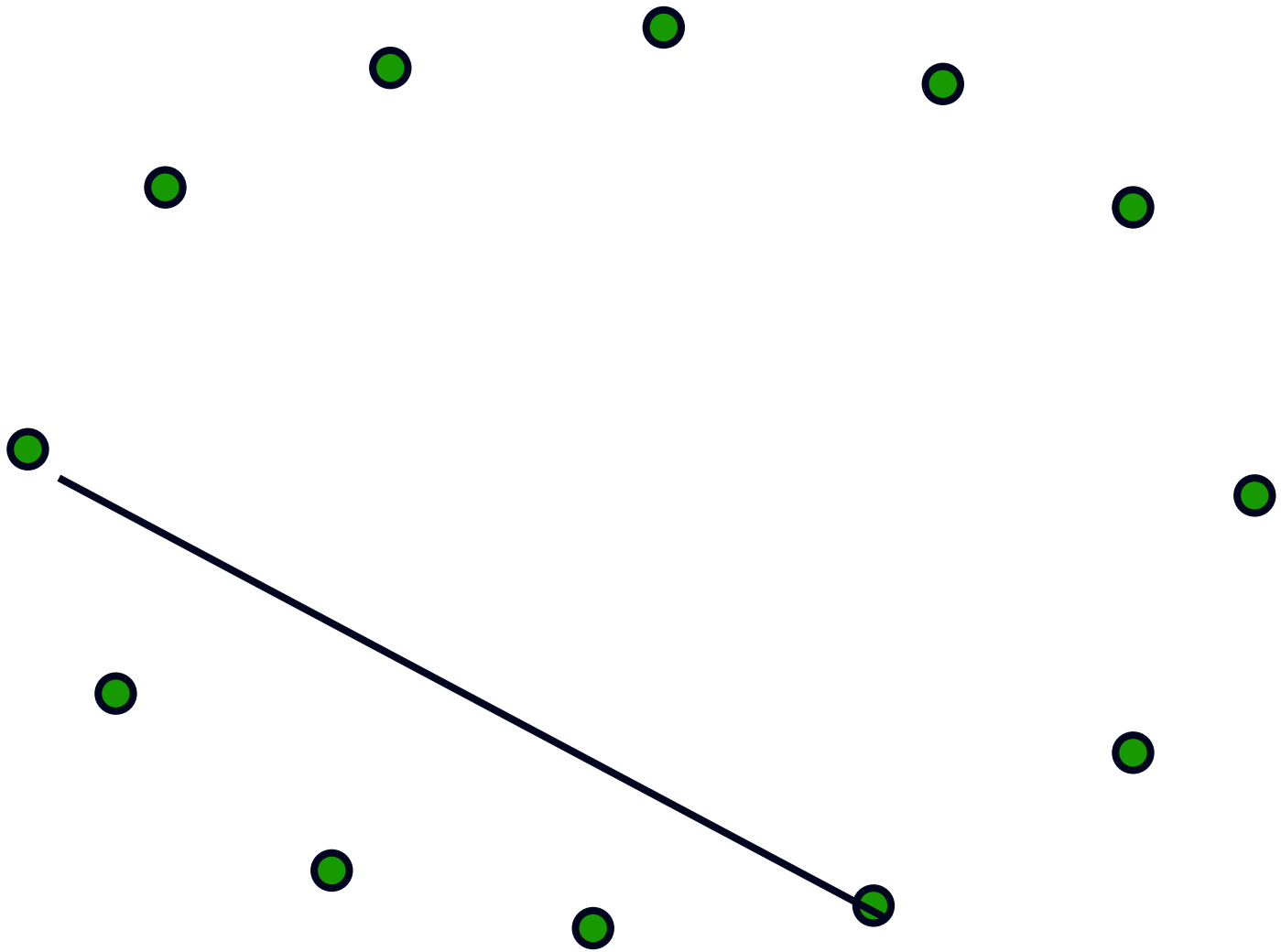

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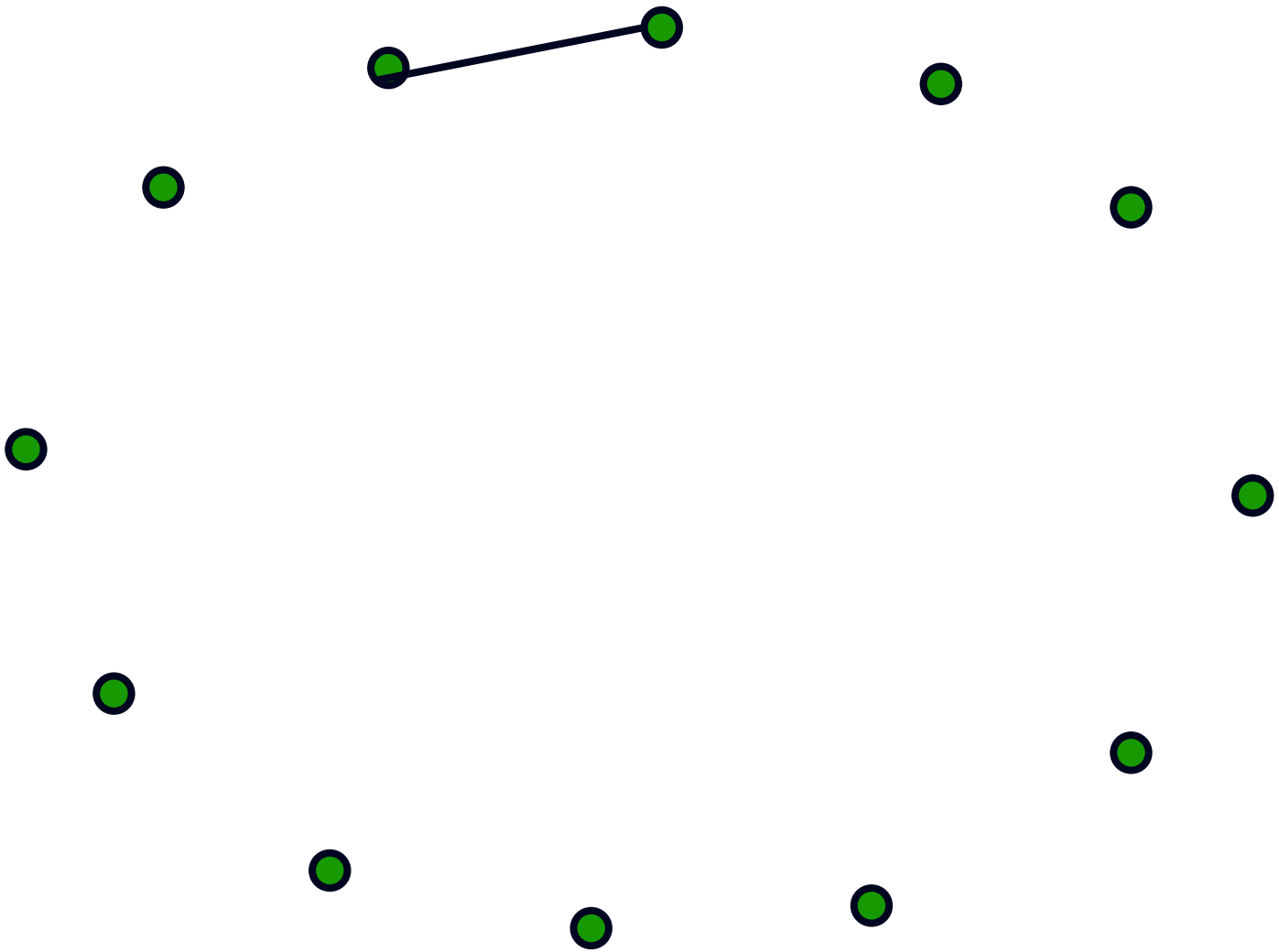

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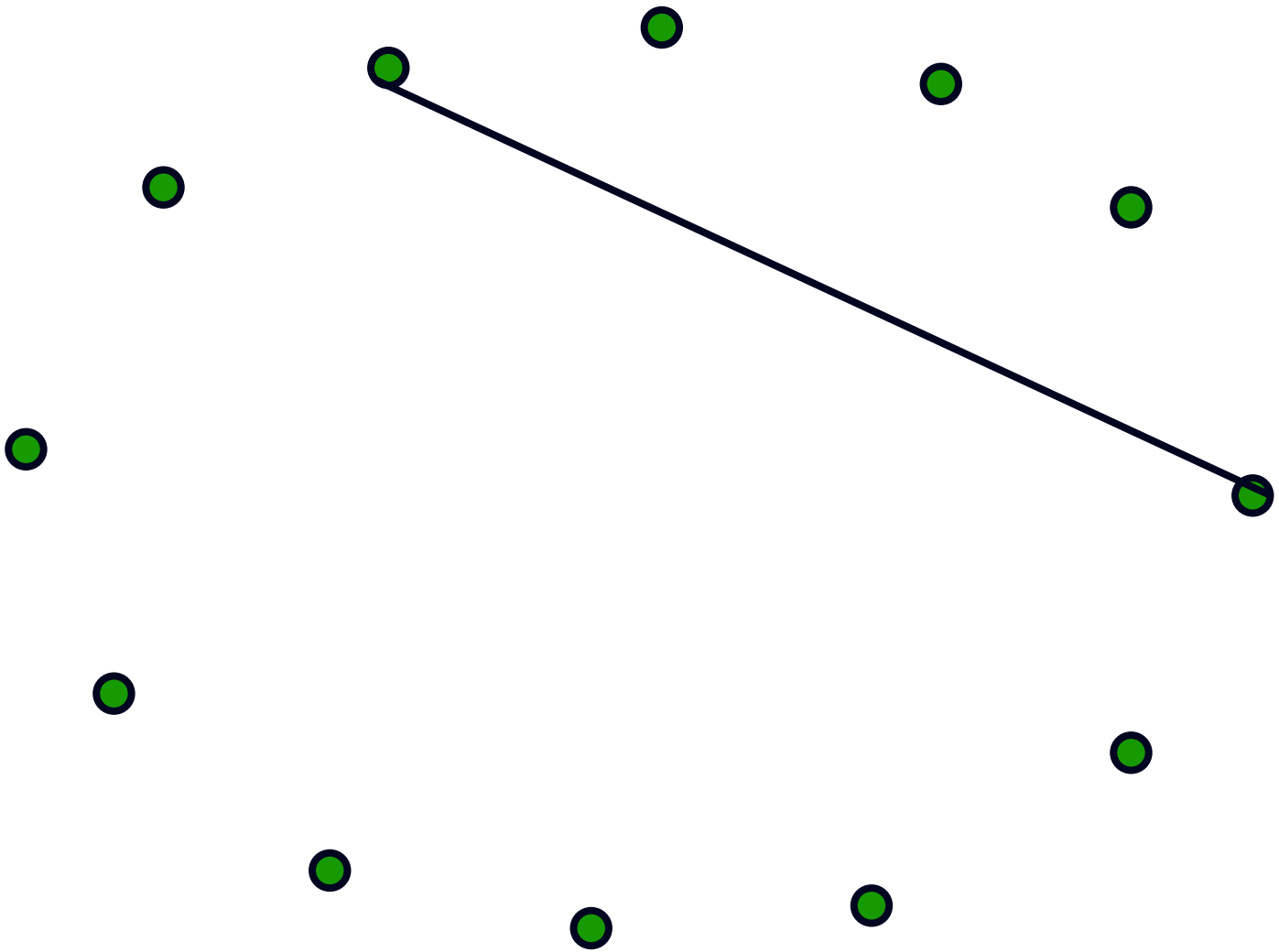

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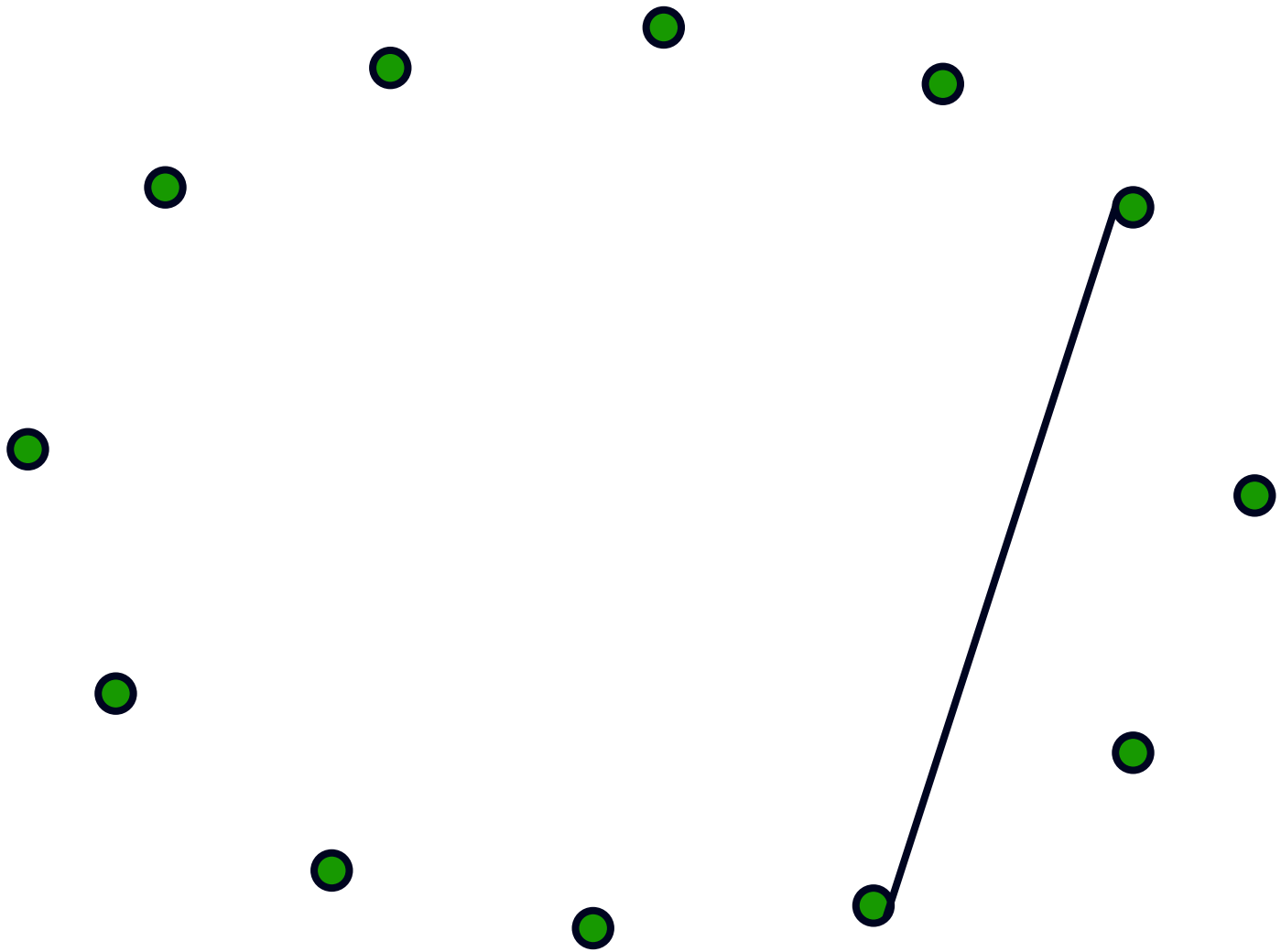

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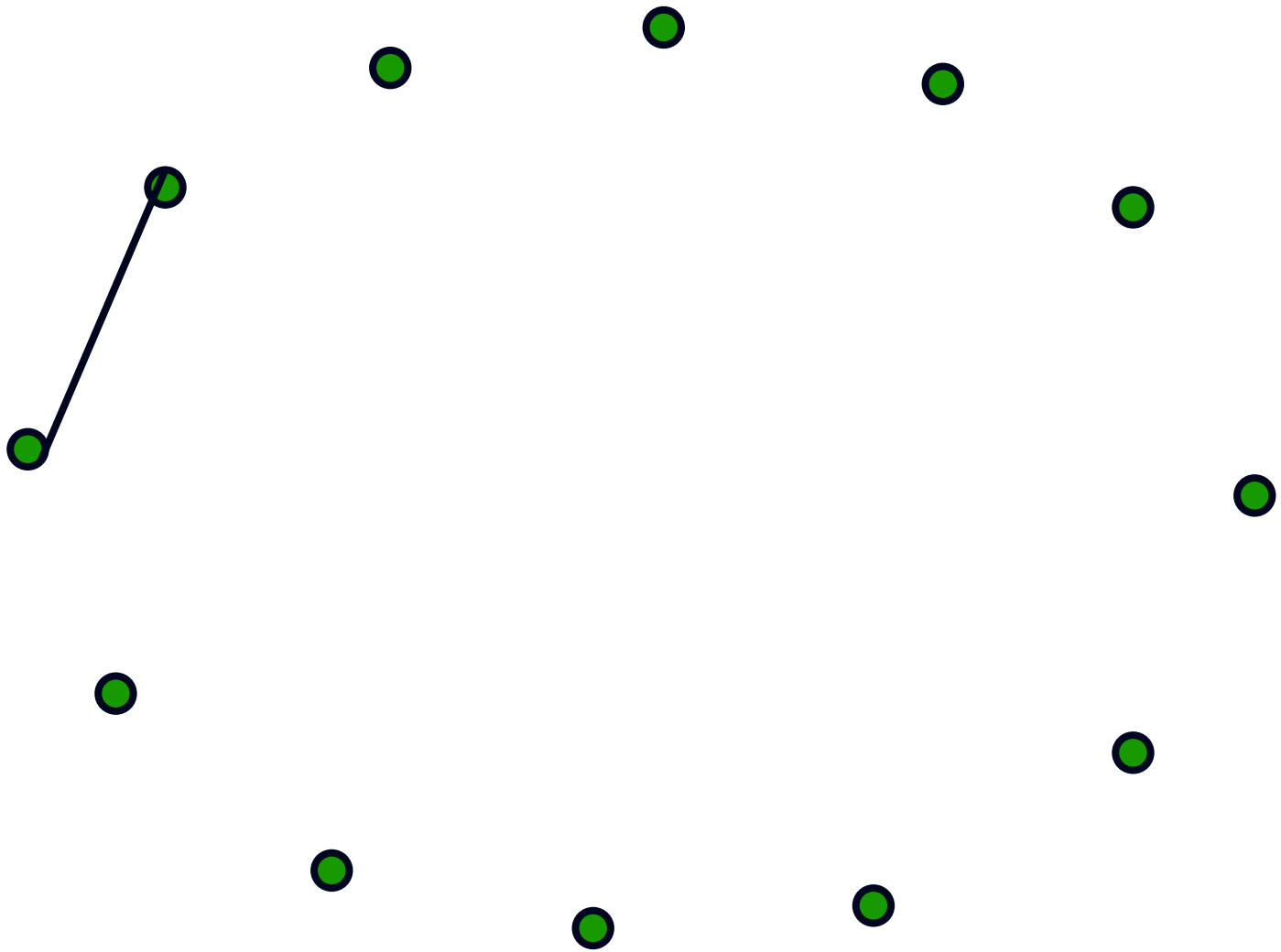

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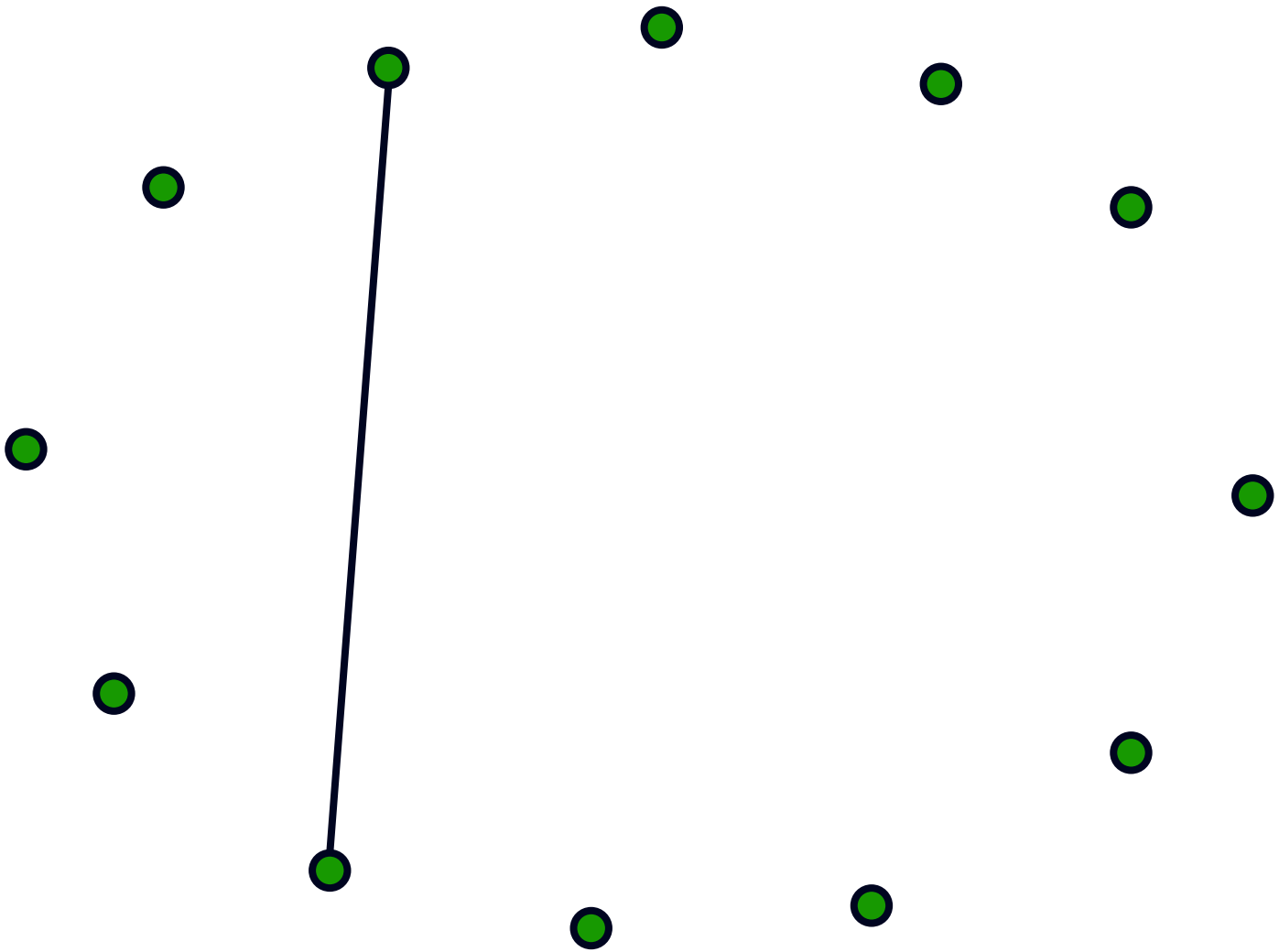

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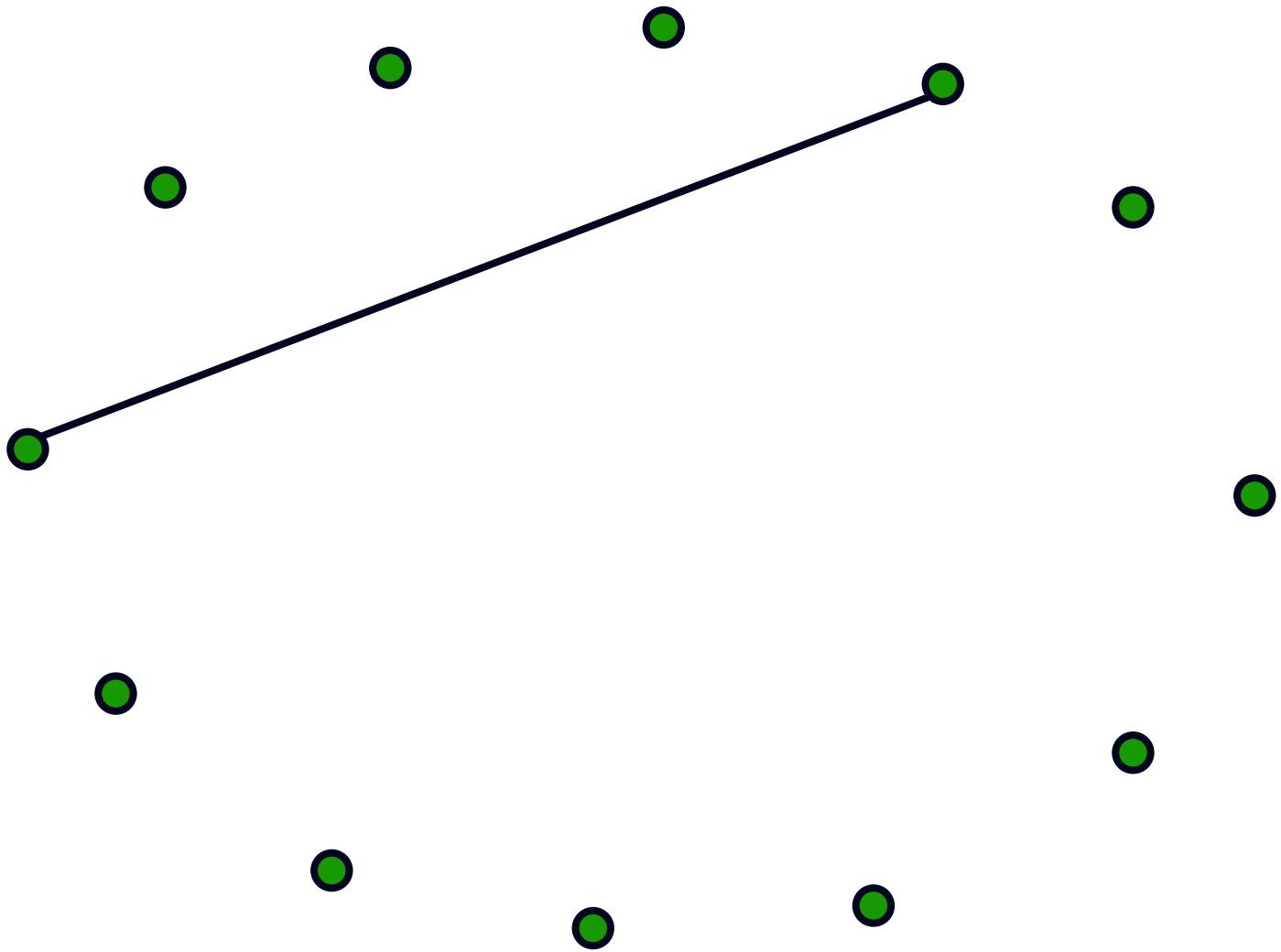

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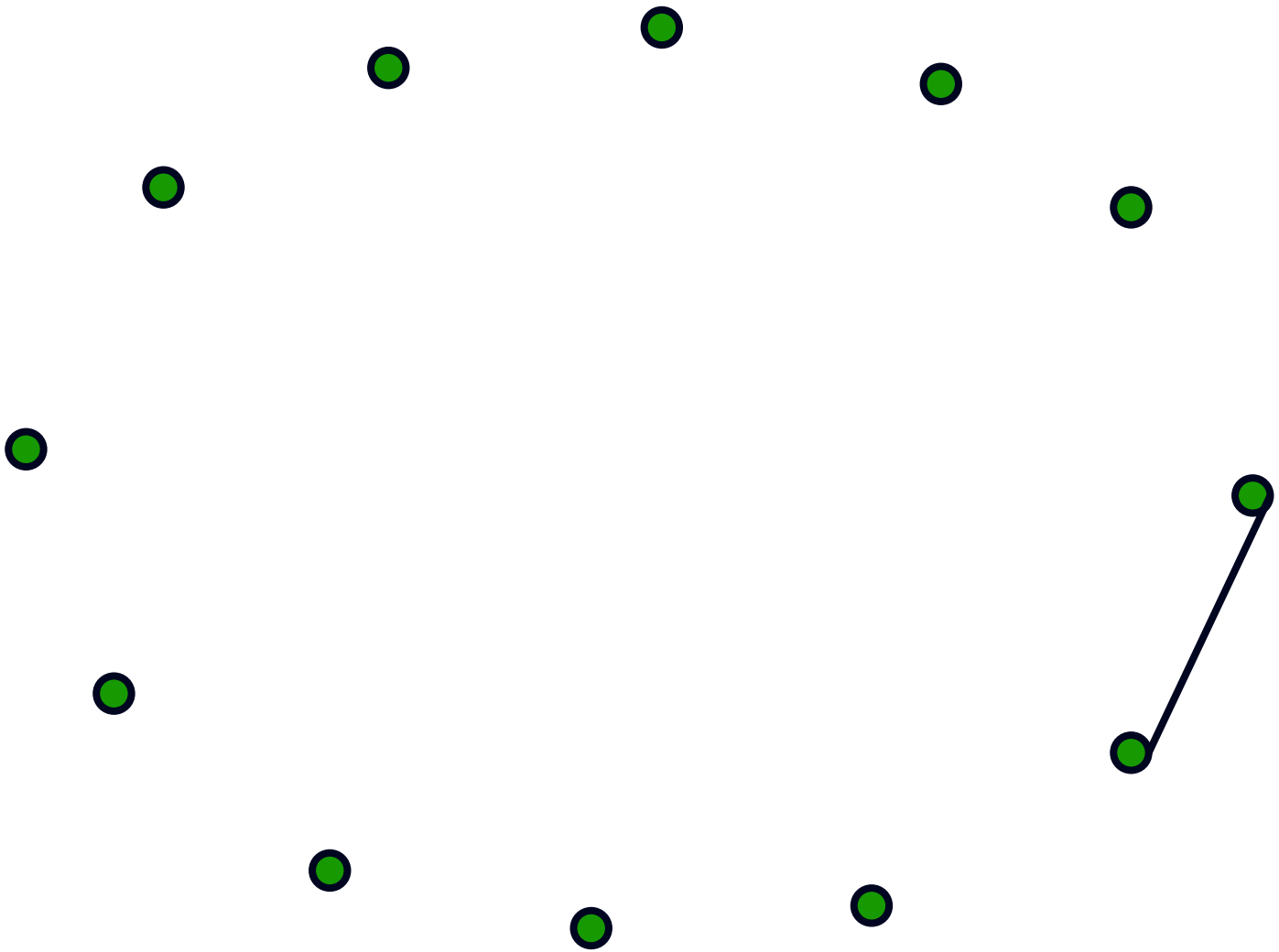

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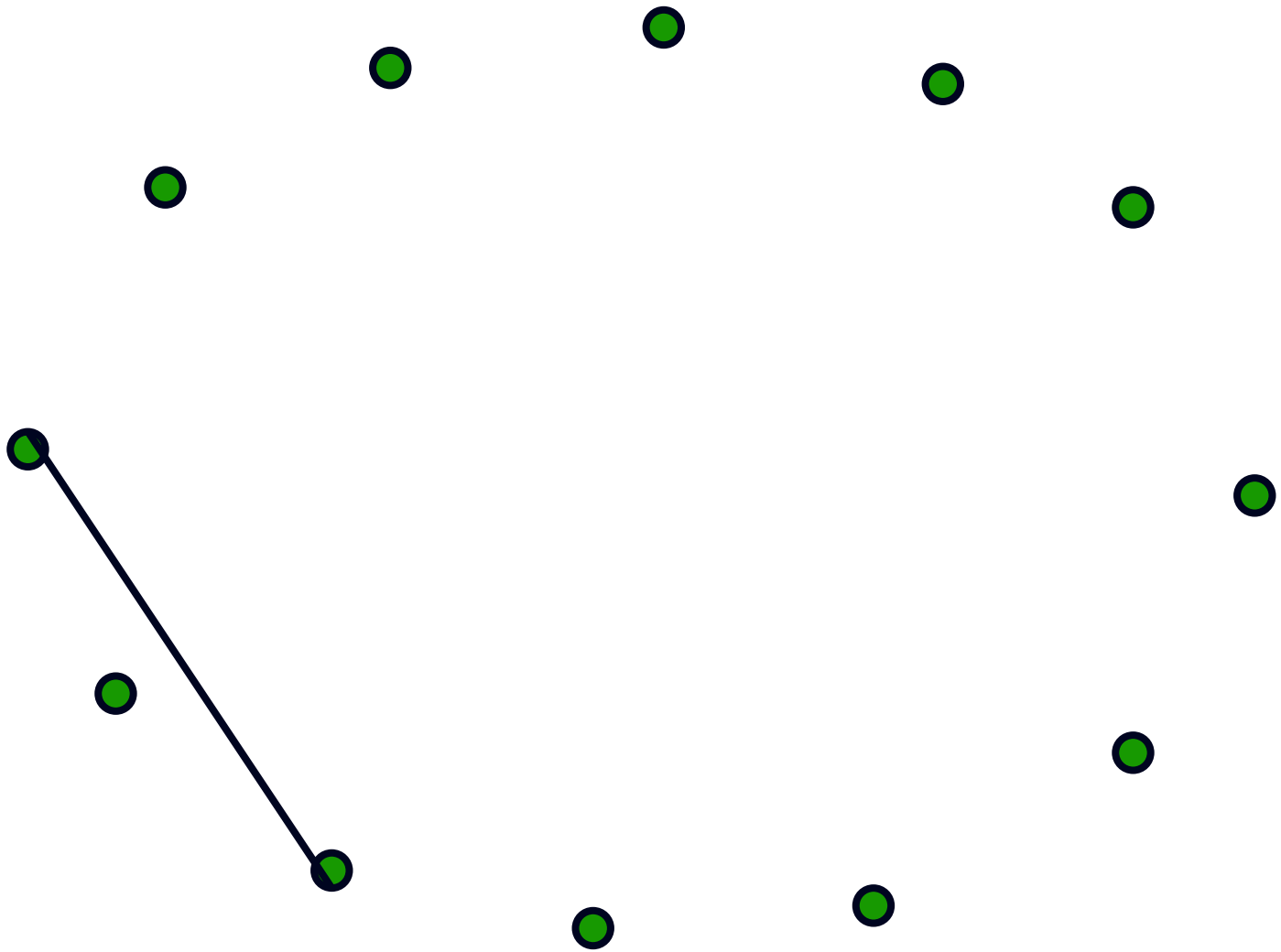

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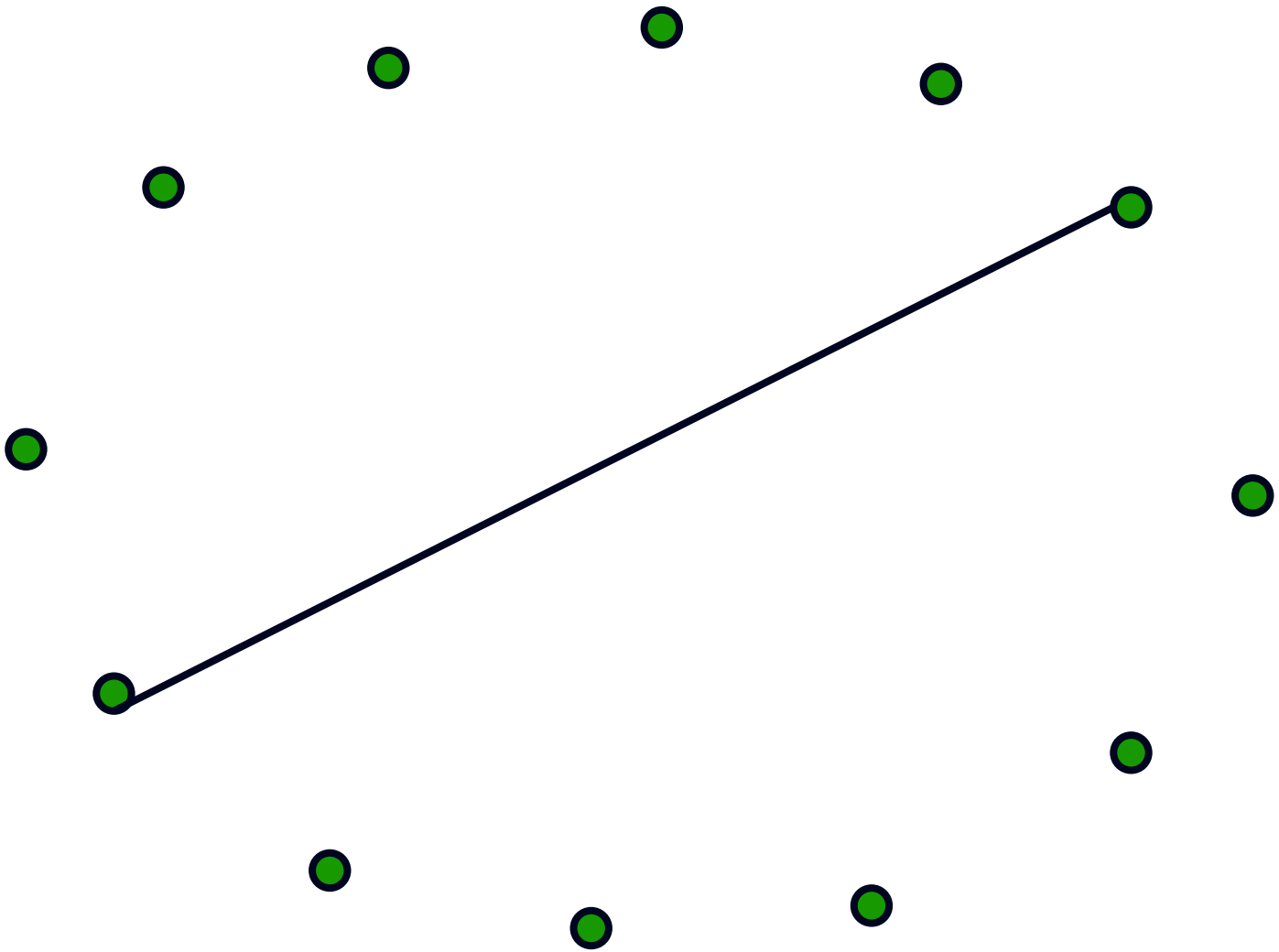

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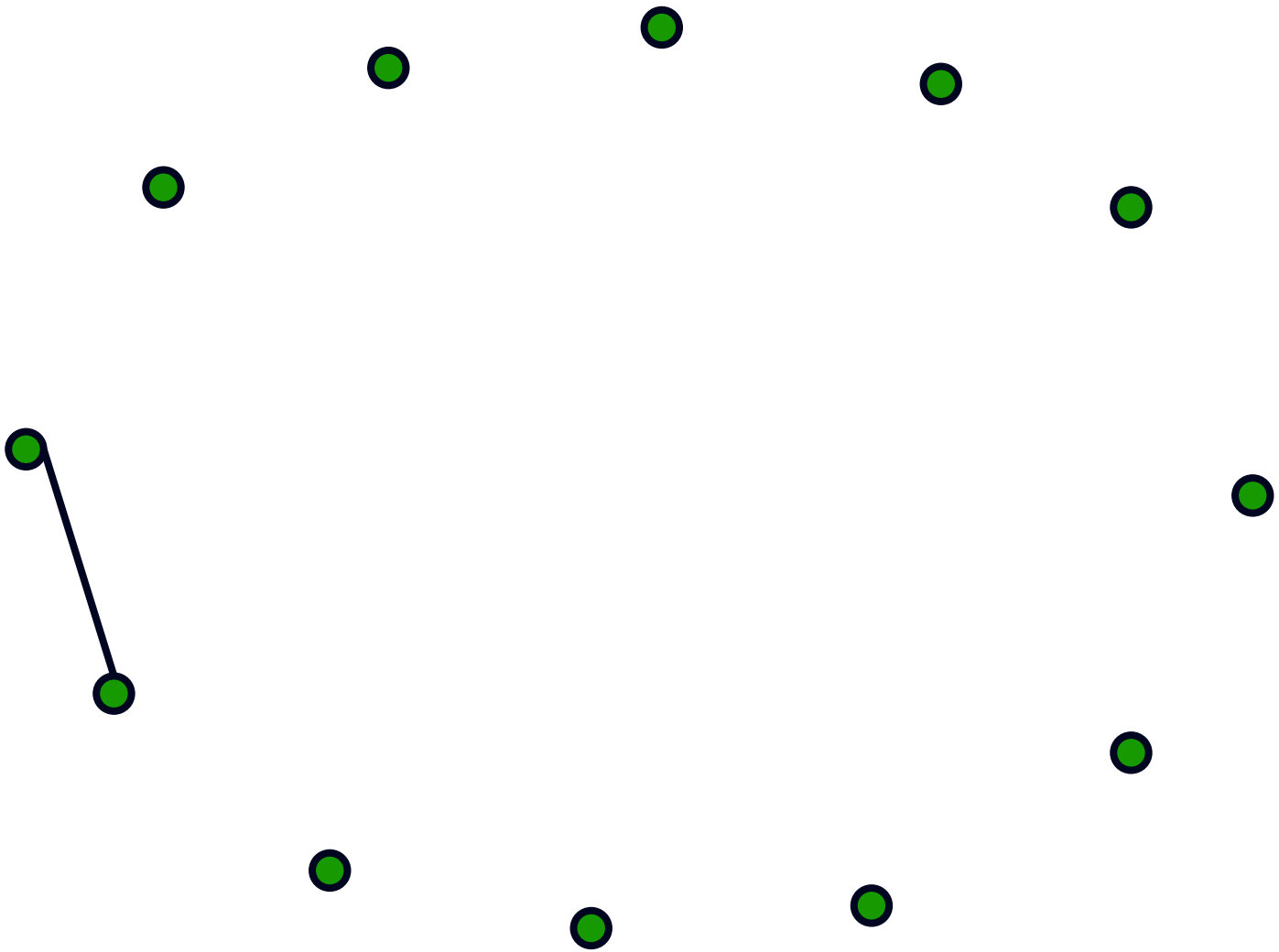

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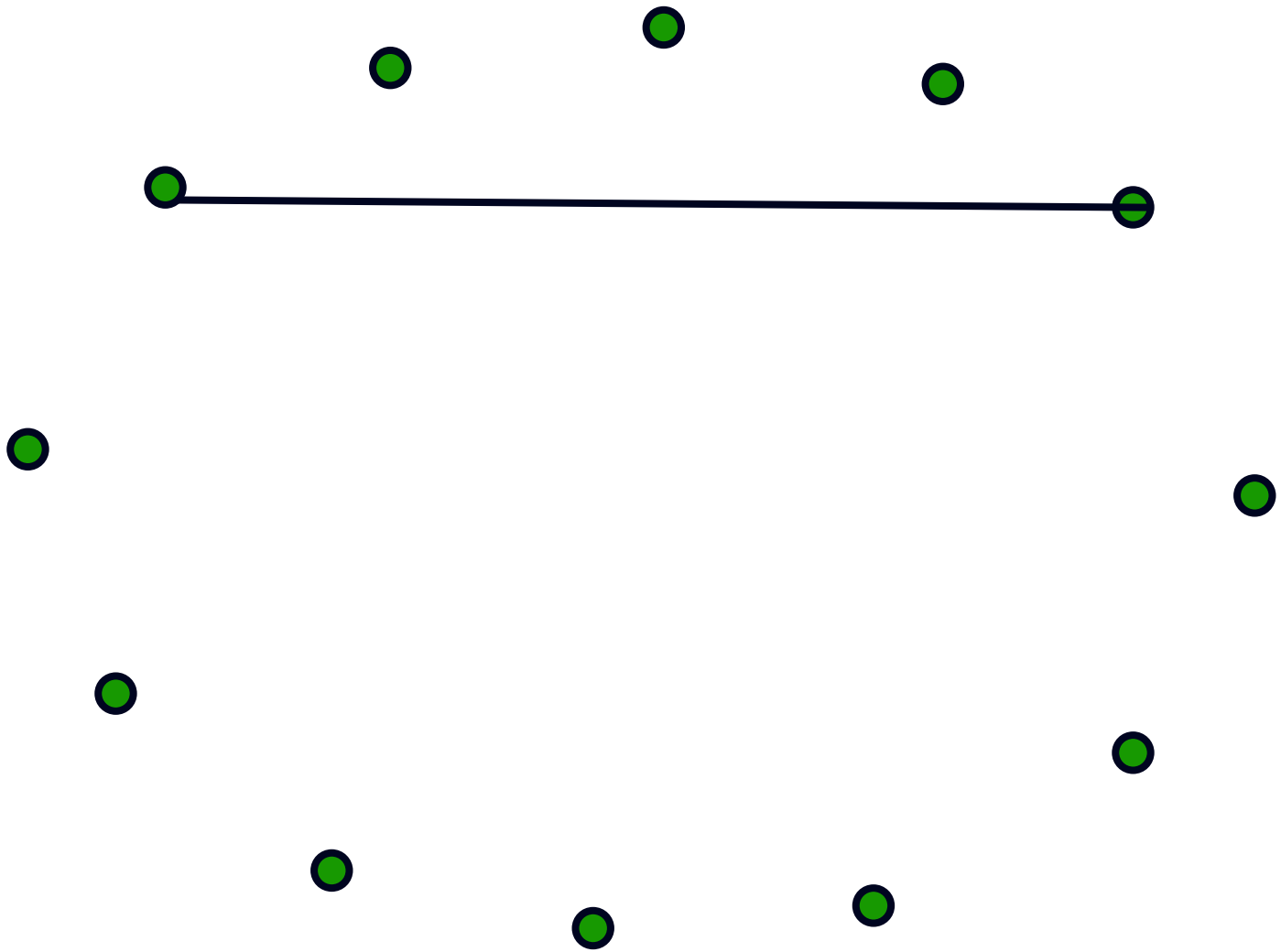

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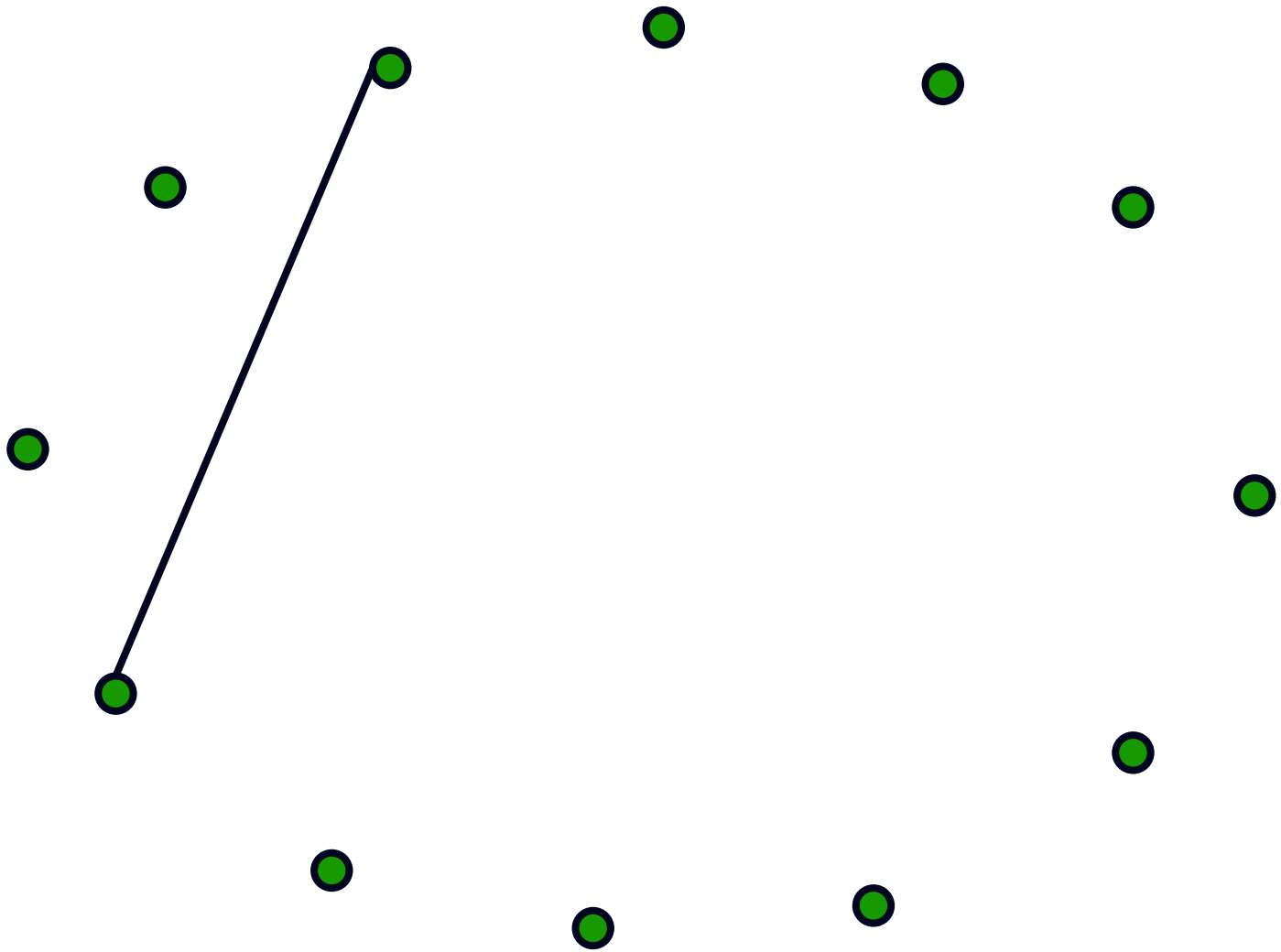

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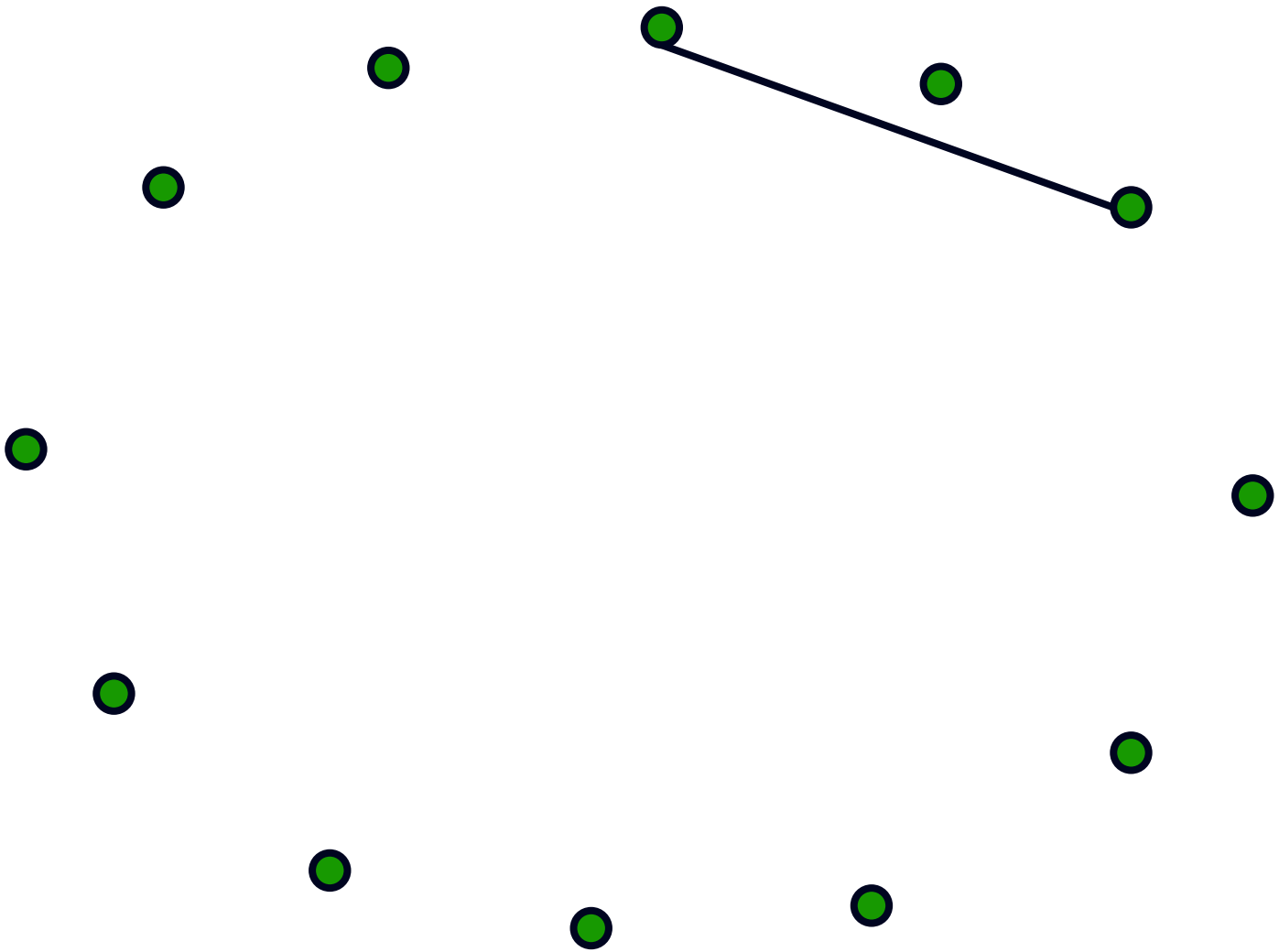

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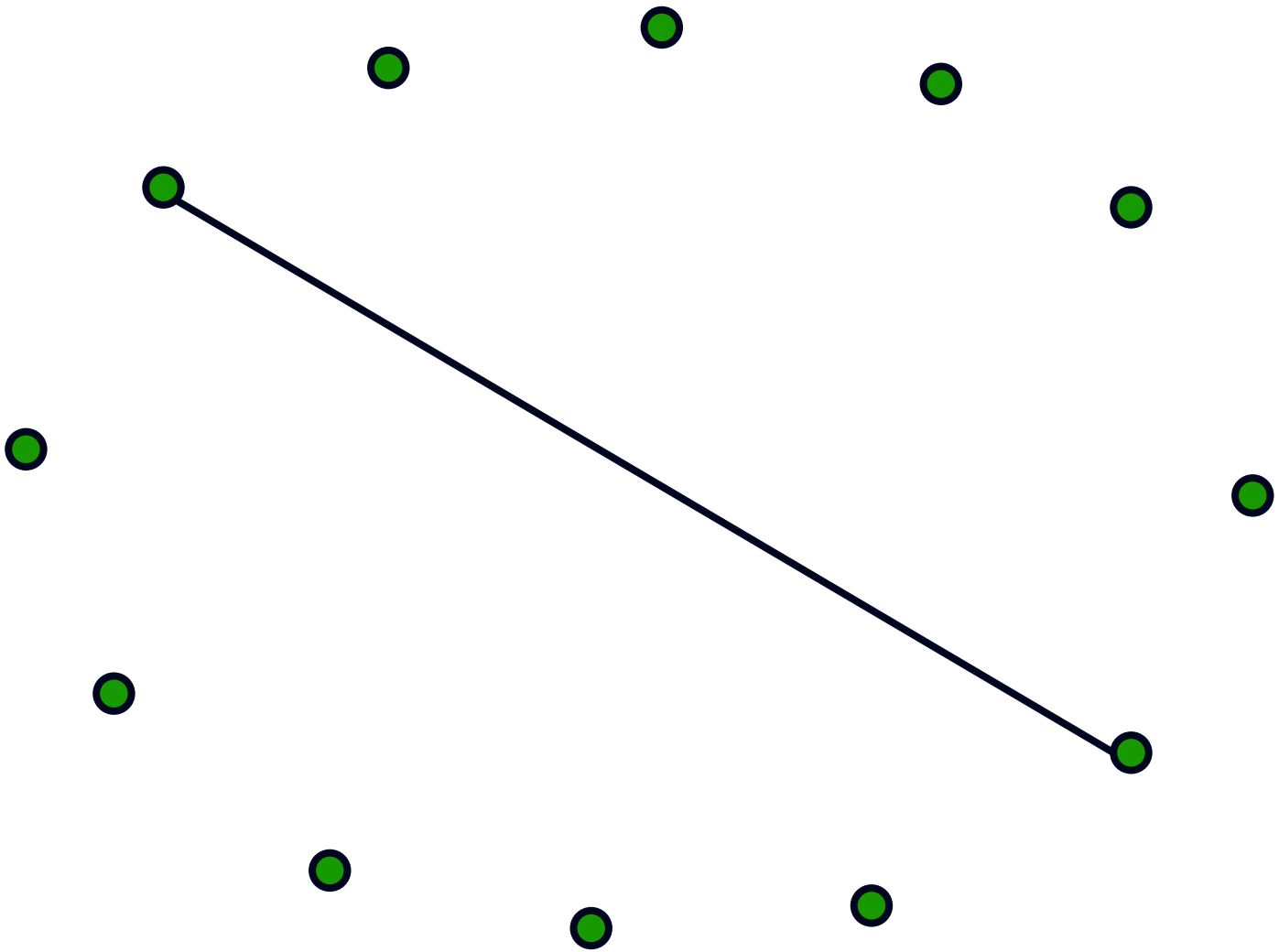

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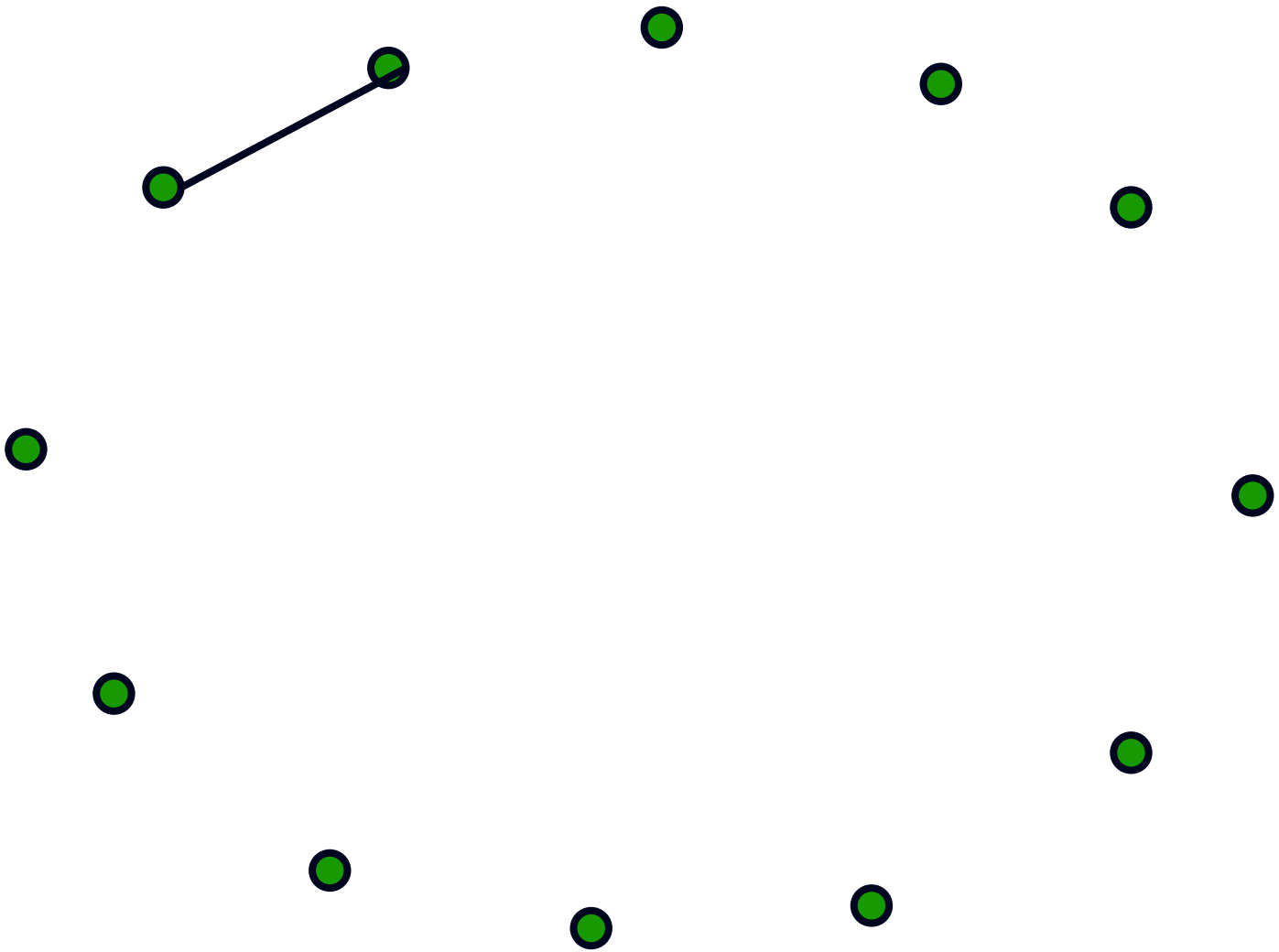

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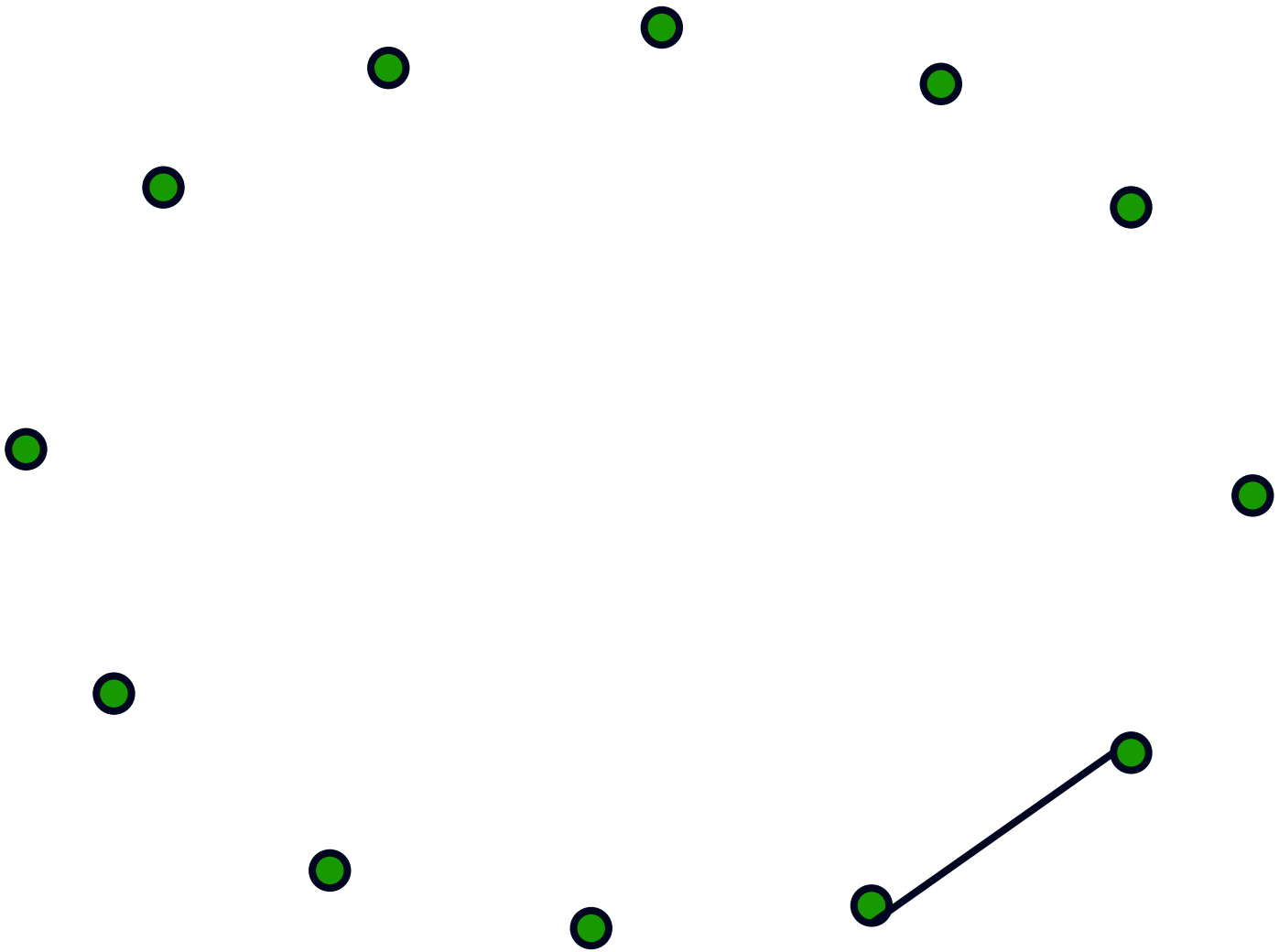

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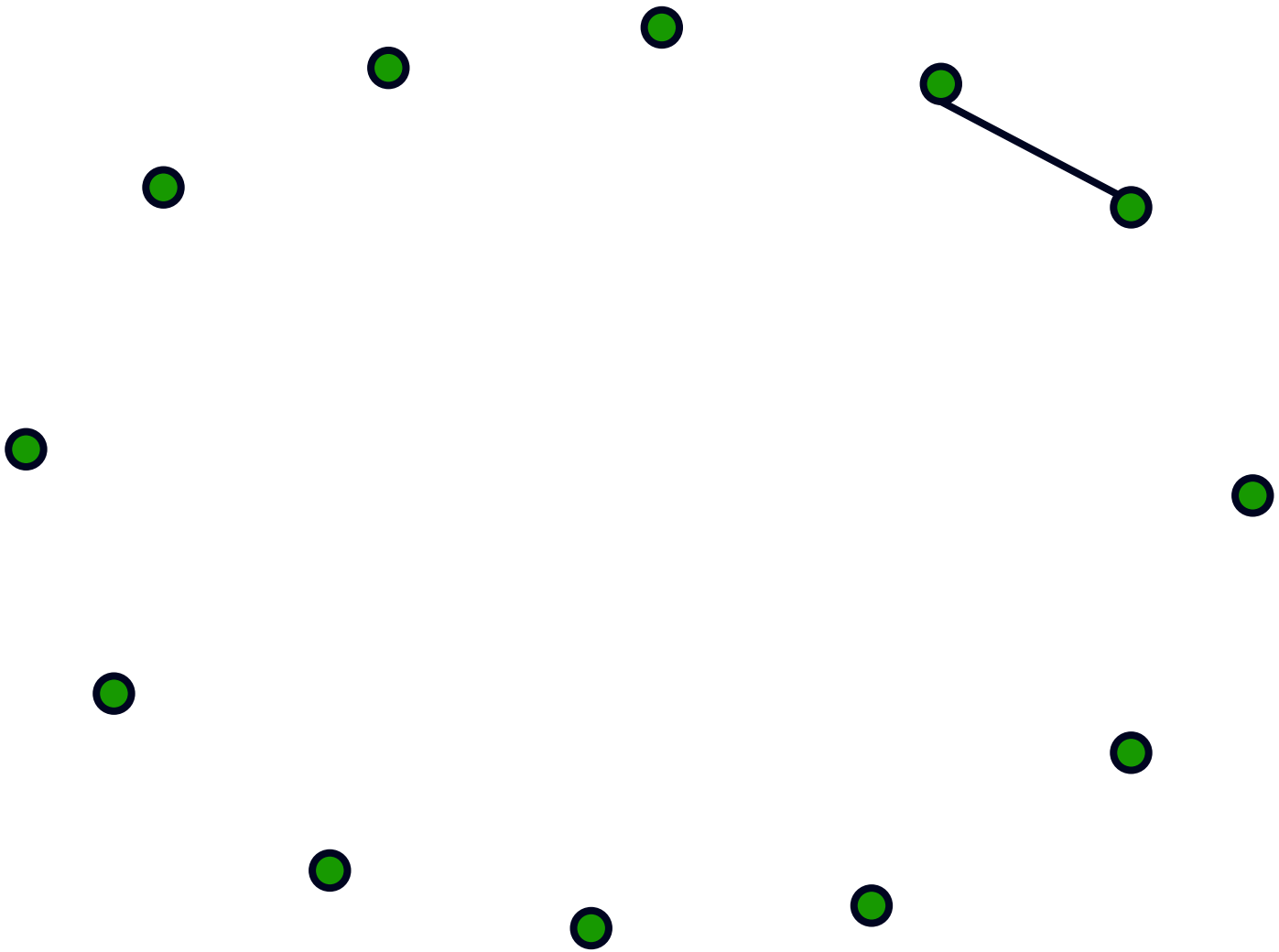

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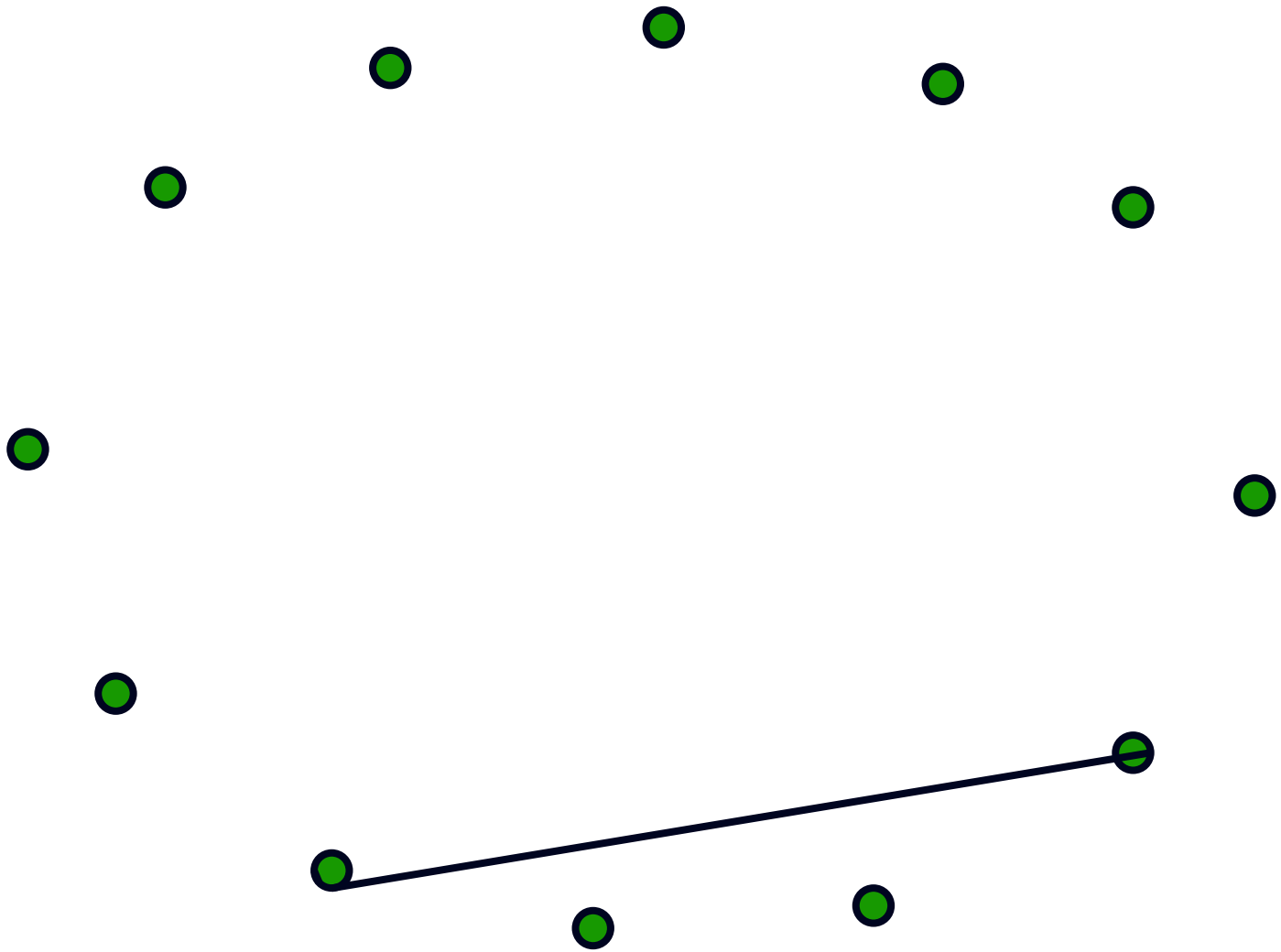

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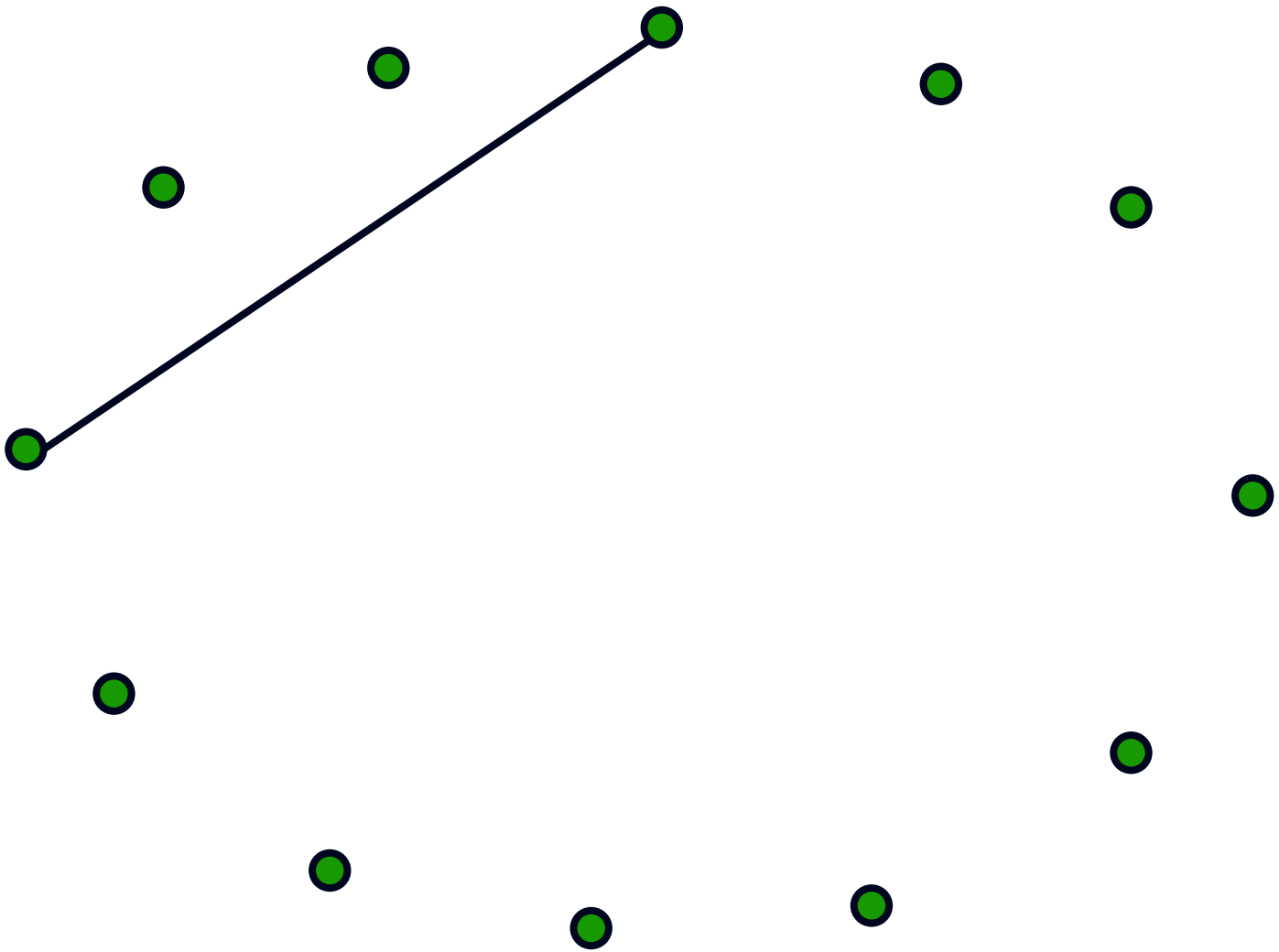

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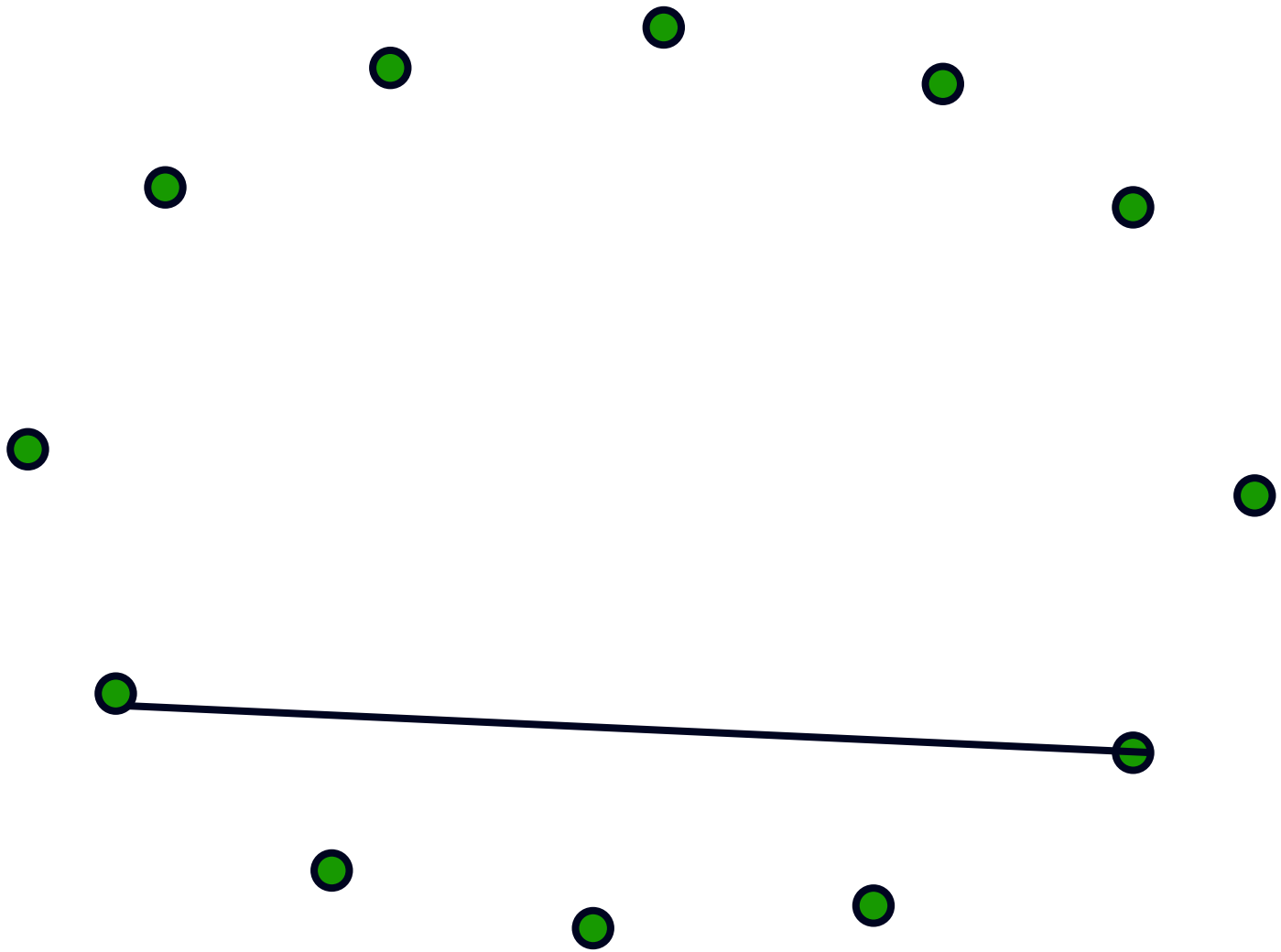

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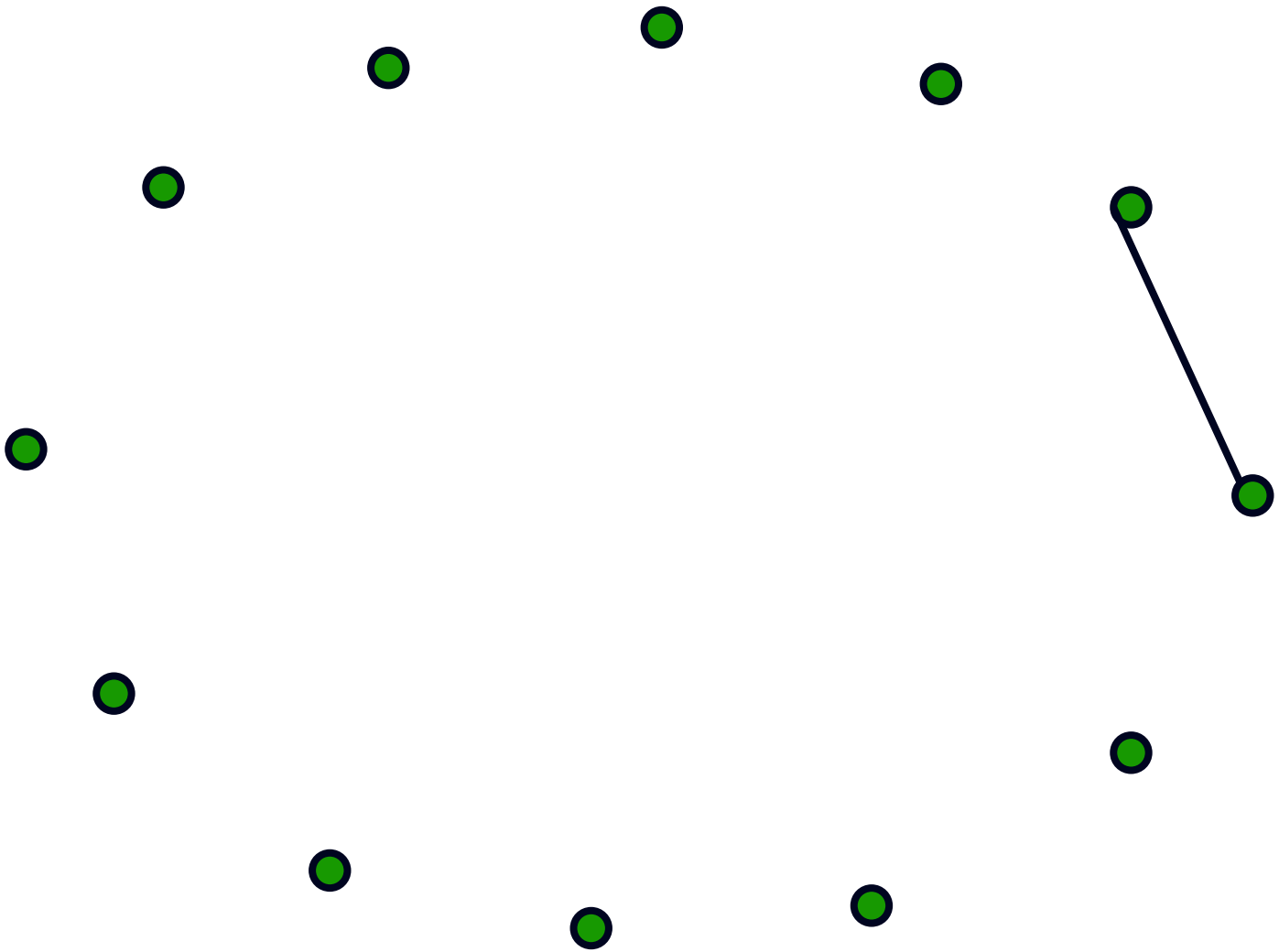

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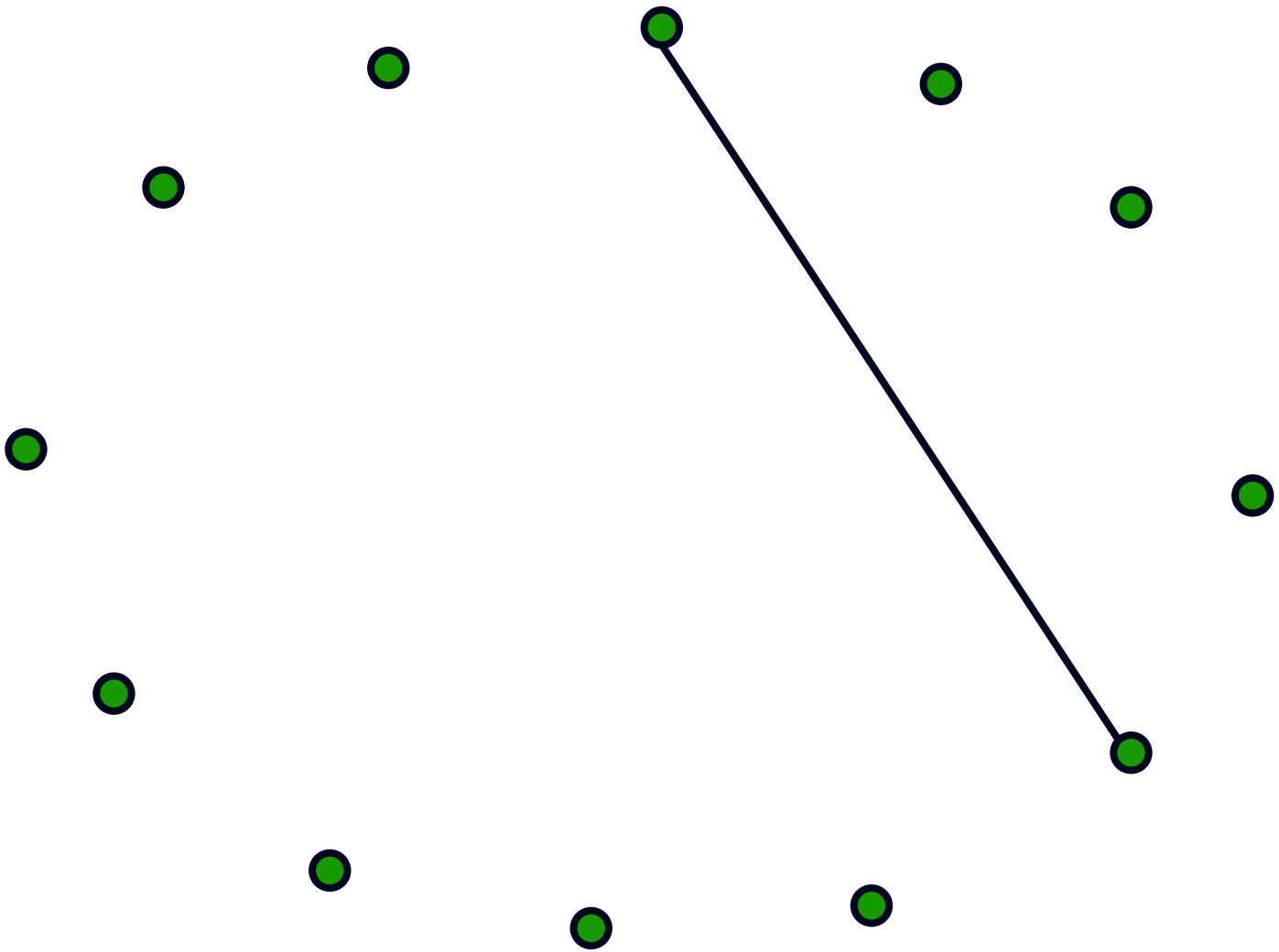
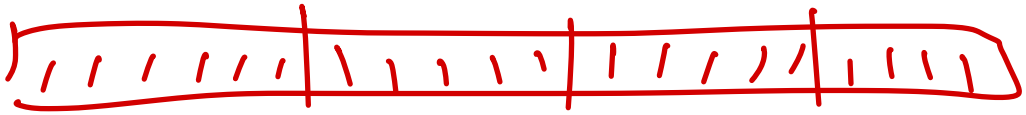


Max Cut Lower Bound



- Hard distributions:
  - YES: Random union of matchings crossing hidden bipartition
  - NO: Random union of matchings.
- Analysis:
  - Divide long stream into O(1) smaller substreams – each substream=matching.
  - Algorithm learns nothing in any single substream ⇐ Boolean Hidden Matching Lower Bound [GKKRW]
  - Hybrid argument to combine O(1) substreams.
  - Yields $\Omega(\sqrt{n})$ space lower bound to beat trivial approximation.
- $\Omega(n)$ lower bound more complex – omitted.

Singer's Thesis

Boolean Hidden Matching Problem

- One-way communication problem.
- Alice gets a random cut on vertex set $[n]$.
- Bob gets a random matching on $[n]$ of size αn along with a 0/1 label on each edge.
  - NO: 0/1 labels random
  - YES: $1 \Rightarrow$ edge crosses cut, $0 \Rightarrow$ doesn't cross.
- Challenge: Alice sends message to Bob, Bob to distinguish YES from NO.
- Lower bound theorem [GKKRW]: $\Omega(\sqrt{n})$ communication required.

GVV+CGV Algorithms for Max DiCut

- Define $Bias(v) \stackrel{\text{def}}{=} indeg(v) - outdeg(v)$
- $\text{Bias}(G) \stackrel{\text{def}}{=} \frac{1}{2}\sum_v |Bias(v)|$
- Claim 1 [GVV]:
  - Bias can be estimated in polylog space. (ℓ_1-norm estimation)
  - $Dicut(G) \leq \frac{Bias(G)+m}{2}$
  - $Bias(G) \leq Dicut(G)$ (Greedy rounding)
  - Output: $\max\left\{\frac{m}{4}, Bias(G)\right\} \Rightarrow$ 2/5-approx.
- Claim 2 [CGV]: $\text{Bias(G)} \leq \frac{m}{3} \Rightarrow Dicut(G) \geq \frac{m}{4} + \frac{Bias(G)^2}{4(m-2Bias(G))}$

(Rand. Rounding w.p. $\frac{1}{2} - \frac{Bias(G)}{2(m-2Bias(G))}$) $\Rightarrow$ 4/9-approx.

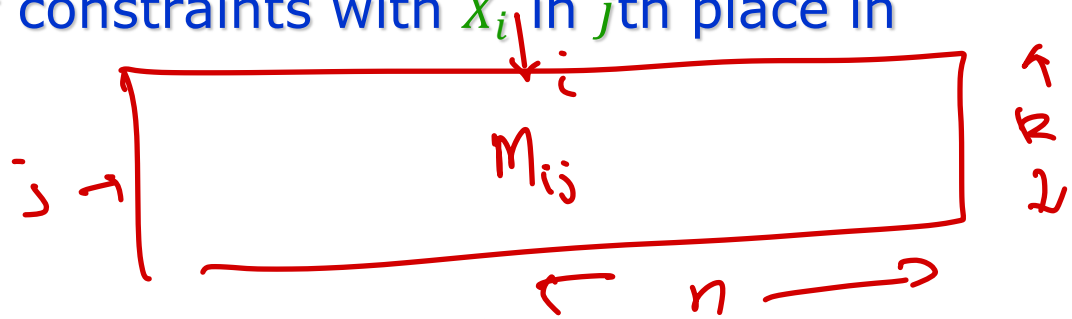
Generalizing to other CSPs: Challenges

- What is bias, say for $f(x, y, z) = x \wedge (y \oplus z)$
  - E.g.: $x \wedge (a_1 \oplus a_2), a_3 \wedge (x \oplus y), y \wedge (x \oplus a_4)$
- Why is ℓ_1-estimation useful in Dicut approximation
  - $\ell_1(x_1, \ldots, x_n) = \max_{a_1 \ldots a_n \in \{-1,1\}} \{\sum_i a_i x_i\}$
  - Useful generalization: $||M||_{\{1,\infty\}} := \max_{a \in [q]^n} \{\sum_i M_{i,a(i)}\}$
  - Computable in polylog space
- Dicut Analysis: Graph theory, some 3-var. calculus, Why did rounding end up optimal?

Dichotomy for Sketching

- Stepping back: Suppose algorithm gets entire "incidence matrix" M (and only this info)

  (M_{ij} = fraction of constraints with X_i in jth place in constraint.)



  - How well can this algorithm perform?

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- Thm: - Alg can be sketched in polylog space;

  - If Alg can't solve then no $o(\sqrt{n})$-sketching alg

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  - Criterion is decidable in finite time.

Decidability & Criterion

- Main insight: Suffices to look at instances on kq variables.
- Instance = distribution of constraints over kq variables.
- …
- $\exists$ convex sets K_γ^Y and $K_\beta^N \subseteq \Delta([q])^k \subseteq \mathbb{R}^{qk}$ s.t. Algorithm exists if $K_\gamma^Y \cap K_\beta^N = \emptyset$

Complexity?

$\{01, 10\}$ $\{00, 11\}$

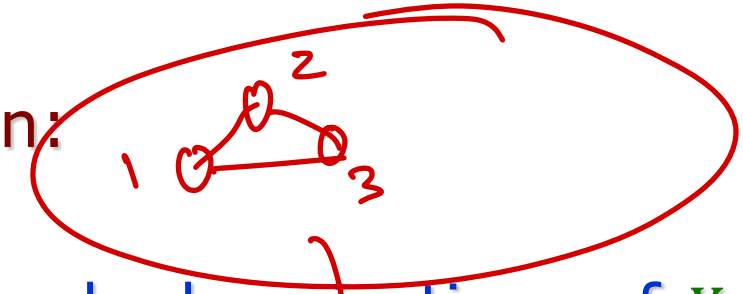
- If Algorithm does not exist, then?
  - There exist distributions D^Y and $D^N \in \Delta([q]^k)$ with matching (one-wise) marginals corresponding to two "interesting" instances.
- Theorem: If such D^Y and D^N exist then $o(\sqrt{n})$ space sketching algorithm can't solve (γ, β)-distinguishability!
- One-wise Theorem: If such D^Y and D^N exist with $D^N = \mathrm{Unif}([q]^k)$ then $o(\sqrt{n})$ space streaming algorithm can't solve (γ, β)-distinguishability!

Extends to Padded One Wise (k=q=2):

$D^Y = \tau D_0 + (1-\tau) D_1; D^N = \tau D_0 + (1-\tau)\mathrm{Unif}([q]^k)$

Underlying Communication Problem

- Randomized Mask Detection:
  - Alice $\leftarrow X \sim \mathrm{Unif}([q]^n)$
  - Bob $\leftarrow$ Seq. of disjoint masked projections of X
    - Projection: $X \to (S, X|_S) \quad S \subseteq [n],\ |S| = k$
    - Masked: $(S, X|_S) \to \left(S, X|_S + m \ (mod\ q)\right)$
      - YES: $m \sim D^Y$; NO: $m \sim D^N$
    - Sequence: $S_1, \ldots, S_{.0001n}$
    - Disjoint: $S_i \cap S_j = \emptyset$
    - Challenge: Distinguish YES from NO
- Thm: Requires $\Omega(\sqrt{n})$ bits Alice $\to$ Bob if marginals of D^Y and D^N match.



D^Y

$b_1\ b_2\ b_3 \to$

D^N

$1 \oplus b_1$

$2 \oplus b_2$

$3 \oplus b_3$

Proof ingredients

- Step 1: If $D_N = Unif([q]^k)$ then (D_Y, D_N)-RMD hard.
  - Proof extends Fourier analytic proof of GKKRW.
- Step 1.5: $\left(\mathrm{Unif}(\{u,v\}), \mathrm{Unif}(\{\max(u,v), \min(u,v)\})\right)$-RMD hard
- Step 2: General (D_Y, D_N) :
  - $(D, D - \epsilon\, \mathrm{Unif}(\{u,v\}) + \epsilon\, \mathrm{Unif}(\{\max(u,v), \min(u,v)\})$-RMD hard
  - Can go from D_Y to D_N in $N(q,k)$ steps.

Open Questions

- Sketching = Streaming?
  - Extend sqrt lower bounds from sketching to streaming!
  - Challenge: Walk length algorithm!
- Sqrt = Linear?
  - Is there a dichotomy at linear space?
  - Challenge: Template-based algorithms!
  - Can Dicut approximation be improved with o(n) space?
- $\omega(n)$-lower bounds? (Trivial: $O(n \log n)$ …)

Thank You!