Streaming & Sketching CSPs

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Based on joint works with Chi-Ning Chou, Alexander Golovnev, Noah Singer, Ameya Velingker and Santhoshini Velusamy.
This Talk

- CSPs (& approximation & streaming/sketching)
- Background + Motivation
- Our results
- Some Proof Ideas
Constraint Satisfaction Problems (CSPs)

- Class of infinitely many problems.
**Constraint Satisfaction Problems (CSPs)**

- **Class of infinitely many problems.**

- **Specified by** \( q, k \) **and family** \( F \subseteq \{ f : [q]^k \rightarrow \{0,1\} \} \).

- **Instance of MaxCSP(\( F \)):** \( \Psi = (X_1, ..., X_n; C_1, ..., C_m) \);
  **Constraint** \( C_j(X) = f_j(X_{i_1(j)}, ..., X_{i_k(j)}) ; f_j \in F \)

- \( \text{opt}(\Psi) \equiv \max_{a \in [q]^n} \left\{ \frac{1}{m} \sum_j C_j(a) \right\} \)
Constraint Satisfaction Problems (CSPs)

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Instance of $\text{MaxCSP}(F)$: $\Psi = (X_1, \ldots, X_n; C_1, \ldots, C_m)$; Constraint $C_j(X) = f_j(X_{i_1(j)}, \ldots, X_{i_k(j)}); f_j \in F$

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Examples:
- $\text{MaxCut} (F = \{\oplus\})$
- $\text{MaxDicut} (F = \{x \land \bar{y}\})$
- Max 3SAT ($|F| = 11$), Max Exact 3SAT ($|F| = 4$), Max $q$-Colorability.
“Special Case”: Boolean MaxCSP($F$):

- $q = 2$;
- Constraints $C = f(X_1, \overline{X_2}, ..., \overline{X_k})$
  - Constraints applied to literals.

Warning: MaxCSP($F$) $\neq$ Boolean MaxCSP($F$)

$\forall F \exists G$ s.t. Boolean MaxCSP($F$) $=$ MaxCSP($G$).
(converse not true. MaxCut, MaxDicut ...)
Streaming & Approximation

- Streaming input
  - **Streaming:** \( s(n) \)-space algorithm – gets one constraint at a time.
  - **Sketching:** Maintains sketch \( S(\sigma) \) with \( |S(\sigma)| \leq s(n) \);
  - **Restriction:** \( (S(\sigma), S(\tau)) \to S(\sigma \circ \tau) \)
  - **Space milestones:** polylog, sqrt, or (nearly-)linear.

- Approximations:
  - **Usual notion:** \( \alpha \)-approximation:
    - **Output** \( v \) s.t. \( \alpha \cdot \text{opt}(\Psi) \leq v \leq \text{opt}(\Psi) \)
  - **Refined notion:** \((\gamma, \beta)\)-distinguishability:
    - **Output** Yes if \( \text{opt}(\Psi) \geq \gamma \), No if \( \text{opt}(\Psi) \leq \beta \)
  - **Equivalence:** \( \alpha = \min_{\gamma} \max_{\beta} \frac{\beta}{\gamma} \)
Trivial Approximations:

- $\tilde{O}(n)$-space, $1 - o(1)$-approx.
  - Maintain $\tilde{O}(n)$-constraints. Solve optimally on those using exponential time.

- $O(1)$-space, $\rho_{\text{min}}$-approx.
  - Defn: $\rho_{\text{min}}(F) := \min_{\psi} \{\text{val}_\psi\}$
  - Notes: Usually $\rho_{\text{min}} > 0$ (unless $0 \in F$)
  - For Boolean CSPs (on literals)
    - $\rho_{\text{min}}(f) = 2^{-k} \cdot |f^{-1}(1)|$ (value of random assignment).
    - E.g. $\rho_{\text{min}}(\text{MaxCut}) = \frac{1}{2}$

- Key question: (When) can we do better than trivial?
Why study CSPs

- Contain some problems of direct interest
  - Max Cut, Max Dicut, Max colorability
- Allow possibility of classification!
- Highlight general algorithms
  - Norm approximations (already used)
  - Local Exploration
  - Crude snapshots
- Identify Phenomena:
  - No $2^{\sqrt{\log n}}$-space algorithms?
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Brief History

- 2011 Bertinoro W’shop (P. 45): “We know nothing +/-”
- 2015-19: Lower bounds for MaxCut [KKS, KKS, VKK]
  - Kapralov-Krachun: $\frac{1}{2} + \epsilon$-approximation requires $\Omega(n)$-space (in $n$ vertex graph) (streaming).
- 2017-20: Algorithms for Max DiCut, Max SAT
  - Guruswami-Velingker-Velusamy: DiCut
  - Chou-Golovnev-Velusamy: DiCut, Max SAT
- 2020: Sketching Classification
  - Chou-Golovnev-Velusamy: Classify all Boolean MaxCSP with $k = q = 2$
Our Results

- **[CGSV21]:**
  - Dichotomy for sketching (polylog vs. sqrt)
  - Polylog space algorithms for infinitely many CSPs
  - $\Omega(\sqrt{n})$ space lower bounds for broad classes ("one-wise ind.", "padded one-wise ind.").

- **[CGSVV 21]:** Linear space lower bounds for subclass of "one-wise-ind".
  - Pins approximability of all MaxCSP($F$) to within q-factor (trivial alg vs. linear space).

- **[SSV21]:** No sublinear algorithms for "Ordering CSPs"
Proof Ideas
Max Cut Lower Bound

- Hard distributions:
  - YES: Random union of matchings crossing hidden bipartition
  - NO: Random union of matchings.

- Analysis:
  - Divide long stream into $O(1)$ smaller substreams – each substream = matching.
  - Algorithm learns nothing in any single substream $\iff$ Boolean Hidden Matching Lower Bound [GKKRW]
  - Hybrid argument to combine $O(1)$ substreams.
  - Yields $\Omega(\sqrt{n})$ space lower bound to beat trivial approximation.
  - $\Omega(n)$ lower bound more complex – omitted.
Boolean Hidden Matching Problem

- One-way communication problem.
- Alice gets a random cut on vertex set $[n]$.
- Bob gets a random matching on $[n]$ of size $\alpha n$ along with a 0/1 label on each edge.
  - NO: 0/1 labels random
  - YES: 1 $\Rightarrow$ edge crosses cut, 0 $\Rightarrow$ doesn’t cross.
- Challenge: Alice sends message to Bob, Bob to distinguish YES from NO.
- Lower bound theorem [GKKRW]: $\Omega(\sqrt{n})$ communication required.
GVV+CGV Algorithms for Max DiCut

- Define $Bias(v) \overset{\text{def}}{=} \text{indeg}(v) - \text{outdeg}(v)$
- $Bias(G) \overset{\text{def}}{=} \frac{1}{2} \sum_v |Bias(v)|$
- Claim 1 [GVV]:
  - Bias can be estimated in polylog space. ($\ell_1$-norm estimation)
  - $Dicut(G) \leq \frac{Bias(G) + m}{2}$
  - $Bias(G) \leq Dicut(G)$ (Greedy rounding)
- Output: $\max \left\{ \frac{m}{4}, Bias(G) \right\} \Rightarrow 2/5$-approx.

- Claim 2 [CGV]: $Bias(G) \leq \frac{m}{3} \Rightarrow Dicut(G) \geq \frac{m}{4} + \frac{Bias(G)^2}{4(m - 2Bias(G))}$
  - (Rand. Rounding w.p. $\frac{1}{2} - \frac{Bias(G)}{2(m - 2Bias(G))} \Rightarrow 4/9$-approx.)
Generalizing to other CSPs: Challenges

- What is bias, say for \( f(x, y, z) = x \land (y \oplus z) \)
  - E.g.: \( x \land (a_1 \oplus a_2), a_3 \land (x \oplus y), y \land (x \oplus a_4) \)

- Why is \( \ell_1 \)-estimation useful in Dicut approximation
  - \( \ell_1(x_1, \ldots, x_n) = \max_{a_1 \ldots a_n \in \{-1,1\}} \{\sum_i a_i x_i\} \)
  - Useful generalization: \( ||M||_{1,\infty} := \max_{a \in [q]^m} \{\sum_i M_{i,a(i)}\} \)

- Computable in polylog space

- Dicut Analysis: Graph theory, some 3-var. calculus, Why did rounding end up optimal?
Algorithms - 1

- Abstracting GVV,CGV algorithm (|F| = 1)
  - For every $i \in [n]$ maintain (non-normalized) distribution $D_i$ over $[q]$.
  - In the end, round using max. likelihood over $D_i$

- Update?
  - If $X_i$ appears as $j$th var in constraint, need to perturb $D_i$. How?

- Idea:
  - (Initially) Guess & fix some $\lambda_{j\sigma}$, $j \in [k], \sigma \in [q]$
  - (Update on $(i,j)$) Add $\lambda_{j\sigma}$ to $D_i(\sigma)$ for every $\sigma \in [q]$
**Algorithms - 2**

- **Streaming implementation?**
  - Can’t output rounding of $D_i$ for $i \in [n]$
  - But can compute $\frac{1}{m} \sum_i \max_\sigma D_i(\sigma)$

  - ((1, $\infty$) norm of $D \in \mathbb{R}^{n \times q}$. [Andoni Krauthgamer Onak])

- **What $\lambda_j \sigma$ to use?**
  - Don’t know! (e.g., $f(x_1, ..., x_k) = \text{sign}(\sum_j (-1.1)^j x_j)$)

- **“Theorem”: Algorithm works if $\exists \lambda_j \sigma, \tau$ s.t.**

  - $\text{opt}(\Psi) \geq \gamma \Rightarrow \frac{1}{m} \sum_i \max_\sigma D_i(\sigma) \geq \tau$
  - $\text{opt}(\Psi) \leq \beta \Rightarrow \frac{1}{m} \sum_i \max_\sigma D_i(\sigma) < \tau$

- **Theorem [CGSV]:** Criterion above is decidable given $F$. 

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Decidability & Criterion

- **Main insight:** Suffices to look at instances on $kq$ variables.
- **Instance:** Distribution of constraints over $kq$ variables.
- ...
- $\exists$ convex sets $K^Y_Y$ and $K^N_\beta \subseteq \Delta([q])^k \subseteq \mathbb{R}^{qk}$ s.t. Algorithm exists if $K^Y_Y \cap K^N_\beta = \emptyset$
Complexity?

- If Algorithm does not exist, then?
  - There exist distributions $D_Y$ and $D_N \in \Delta([q]^k)$ with matching (one-wise) marginals corresponding to two “interesting” instances.

- Theorem: If such $D_Y$ and $D_N$ exist then $o(\sqrt{n})$ space sketching algorithm can’t solve $(\gamma, \beta)$-distinguishability!

- One-wise Theorem: If such $D_Y$ and $D_N$ exist with $D_N = \text{Unif}([q]^k)$ then $o(\sqrt{n})$ space streaming algorithm can’t solve $(\gamma, \beta)$-distinguishability!

Extends to Padded One Wise (k=q=2):

$$D_Y = \tau D_0 + (1 - \tau)D_1; D_N = \tau D_0 + (1 - \tau)\text{Unif}([q]^k)$$
Underlying Communication Problem

- **Randomized Mask Detection:**
  - **Alice** $\leftarrow X \sim \text{Unif}([q]^n)$
  - **Bob** $\leftarrow$ Seq. of disjoint masked projections of $X$
    - **Projection**: $X \rightarrow (S, X|_S)$ $S \subseteq [n]$, $|S| = k$
    - **Masked**: $(S, X|_S) \rightarrow (S, X|_S + m (mod q))$
      - **YES**: $m \sim D^Y$; **NO**: $m \sim D^N$
    - **Sequence**: $S_1, ..., S_{0001n}$
    - **Disjoint**: $S_i \cap S_j = \emptyset$
    - **Challenge**: Distinguish **YES** from **NO**
  - **Thm**: Requires $\Omega(\sqrt{n})$ bits Alice $\rightarrow$ Bob if marginals of $D^Y$ and $D^N$ match.
Open Questions

- Sketching = Streaming?
  - Extend sqrt lower bounds from sketching to streaming!
  - Challenge: Walk length algorithm!

- Sqrt = Linear?
  - Is there a dichotomy at linear space?
  - Challenge: Template-based algorithms!
  - Can Dicut approximation be improved with o(n) space?
  - $\omega(n)$-lower bounds? (Trivial: $O(n \log n)$ ...)

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Thank You!