

Connectivity Properties of Matroids

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Abstract

The bases-exchange graph of a matroid is the graph whose vertices are the bases of the matroid, and two bases are connected by an edge if and only if one can be obtained from the other by the exchange of a single pair of elements.

In this paper we prove that a matroid is “connected” if and only if the “restricted bases-exchange graph” (the bases-exchange graph restricted to exchanges involving only one specific element e) is connected. This provides an alternative definition of matroid connectivity. Moreover, it shows that the connected components of the restricted bases-exchange graph satisfy a “ratios-condition”, namely, that the ratio of the number of bases containing e to the number of bases not containing e is the same for each connected component of the restricted bases-exchange graph. We further show that if a more general ratios-condition is also true, namely, that any fraction α of the bases containing e is adjacent to at least a fraction α of the bases not containing e (where α is any real number between 0 and 1), then the bases-exchange graph has the following expansion property : “For any bipartition of its vertices, the number of edges incident to both partition classes is at least as large as the size of the smaller partition”. In fact, this was our original motivation for studying matroid connectivity, since such an expansion property yields efficient randomized approximation algorithms to count the number of bases of a matroid [18].

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1 Introduction

The *bases-exchange graph* of a matroid was introduced by Edmonds [7] as the graph whose vertex-set is the collection of bases of the matroid, and two bases B, B' are connected by an edge if and only if their symmetric difference has cardinality exactly 2, i.e., B' can be obtained from B by the fundamental exchange operation $B' = B \setminus \{x\} \cup \{y\}$. The *e -restricted bases-exchange graph* is the bipartite subgraph of the bases-exchange graph where only edges involving a particular element e are considered ($B' = B \setminus \{e\} \cup \{y\}$, or $B' = B \setminus \{x\} \cup \{e\}$). Henceforth, for a matroid $\mathcal{M}(S, \mathcal{B})$, where S is the ground-set and \mathcal{B} is the collection of bases, $G(\mathcal{M})$ will denote the bases-exchange graph of \mathcal{M} , and $G_e(\mathcal{M})$ will denote the e -restricted bases-exchange graph of \mathcal{M} . In this paper we focus on certain non-trivial connectivity and expansion properties of $G(\mathcal{M})$ and $G_e(\mathcal{M})$ (roughly, connectivity in matroids corresponds to biconnectivity in graphs: a matroid is connected if and only if there exists a circuit through every pair of elements of its ground-set).

We show that a matroid is connected if and only if the e -restricted bases-exchange graph $G_e(\mathcal{M})$ is connected (Theorem 4.2), thus obtaining a combinatorially interesting alternative definition to matroid connectivity. Despite the simplicity of the above statement, its proof involves rather non-trivial technical manipulations. In fact, equivalent definitions of matroid connectivity have always been non-trivial, and have therefore attracted significant attention (e.g. see the work of Whitney [27], Tutte [23] and Harary [9]). Furthermore, Theorem 4.2 provides a concrete characterization of the connected components of $G_e(\mathcal{M})$: in Theorem 4.3 we show that the connected components of $G_e(\mathcal{M})$ correspond exactly to the connected components of the matroid ground-set. We also observe that all connected components of $G_e(\mathcal{M})$ are isomorphic.

All the above suggest that the following ratios-condition is satisfied by the connected components of $G_e(\mathcal{M})$: Let \mathcal{B}_e (resp. $\mathcal{B}_{\bar{e}}$) denote the collection of bases of \mathcal{M} containing a particular element $e \in S$ (resp. not containing e). Let A be a subset of \mathcal{B}_e and let $\Gamma_e(A)$ be the set of vertex neighbors of A in $G_e(\mathcal{M})$. Then, when $A \cup \Gamma_e(A)$ is an entire connected component of $G_e(\mathcal{M})$, it is the case that $|\Gamma_e(A)|/|A| = |\mathcal{B}_{\bar{e}}|/|\mathcal{B}_e|$ (Corollary 4.4).

We further show that if a similar ratios-condition is satisfied for arbitrary subsets A of \mathcal{B}_e , then the bases-exchange graph is a cutset-expander in the sense conjectured by Mihail and Vazirani [18]: “For any bipartition of the vertices of the bases-exchange graph, the number of cutset edges (edges incident to vertices in both partitions) is at least as large as the size of the smaller partition”. In particular, we have isolated the following *Ratios-Criterion* for expansion: “ $G(\mathcal{M})$ is a cutset-expander if for all $A : A \subseteq \mathcal{B}_e, |\Gamma_e(A)|/|A| \geq |\mathcal{B}_{\bar{e}}|/|\mathcal{B}_e|$ ” (Theorem 3.3). We believe that the Ratios-Criterion might be particularly suitable for matroids because it is the core of an inductive reasoning which imposes structure towards arguing about single-step exchanges that involve a specific element e . Matroids satisfy certain interesting properties for single-step exchanges that are not known to hold for sequences of exchanges. For instance, it is known that matroids possess “single-step complementary properties”: For every pair of bases B and B' and for every $x \in B$, there exists $y \in B'$ such

that both $B \setminus \{x\} \cup \{y\}$ and $B' \setminus \{y\} \cup \{x\}$ are bases ([25], Chapter 1). Such a property, coupled with the restriction to exchanges involving a specific element e as required by the Ratios-Criterion, can be potentially used to a single-step analogue of a Jerrum and Sinclair type of argument for expansion [5] [11] [18].

It has been pointed-out by Mihail and Vazirani [18], that if strong expansion properties for the bases-exchange graph in fact hold, then aside from being of remarkable combinatorial interest, they are of fundamental algorithmic significance. A sequence of well known ideas concerning the convergence-rate of random walks on expanders and the equivalence of uniform sampling and approximate counting for self-reducible problems, imply that a positive resolution of the matroid-expansion conjecture yields simple and efficient algorithms to approximately count the number of bases of any matroid, given, say, an independence oracle. In view of recent spectacular approximation results that have been attributed to expansion arguments [5] [6] [11] [13], the work reported here pinpoints new aspects of matroid-theory that are related to connectivity and expansion. We believe that matroids are the next natural candidates for expansion to yield efficient approximation schemes for counting.

The rest of the paper is organized as follows : In Section 2 we review the fundamental concepts concerning matroid theory, random walks, and approximate counting, in order to provide a context for our work. In Section 3 we present the Ratios-Criterion for expansion (Theorem 3.3), and discuss its suitability for the matroid setup. The key underlying structure in the Ratios-Criterion is the e -restricted bases-exchange graph $G_e(\mathcal{M})$. In Section 4 we focus on the connectivity of the e -restricted bases-exchange graph, and show that $G_e(\mathcal{M})$ is connected if and only if the underlying matroid \mathcal{M} is connected (Theorems 4.2 and 4.3). A necessary condition for the e -restricted bases-exchange graph to pass the Ratios Criterion follows (Corollary 4.4).

2 Bases : Enumeration and the Exchange Graph

Definition: A matroid \mathcal{M} over a finite ground-set S is a pair (S, \mathcal{B}) , where \mathcal{B} (the bases of \mathcal{M}) is a collection of subsets of S satisfying :

- (i) All sets B in \mathcal{B} have the same cardinality, and
- (ii) If B and B' are in \mathcal{B} and x is an element of B , then there is some element y of B' such that $B \setminus \{x\} \cup \{y\}$ is in \mathcal{B} .

Definition: A set $I \subseteq S$ is called independent in $\mathcal{M}(S, \mathcal{B})$ if it is a subset of some basis in \mathcal{B} .

The interested reader is referred to the classical textbook of Welsh [25] for a detailed treatment of matroid theory.

Definition: For a matroid $\mathcal{M}(S, \mathcal{B})$ the bases-exchange graph $G(\mathcal{M})$ of \mathcal{M} is a graph whose vertex-set is the set of bases \mathcal{B} , and two bases B and B' are connected with an edge if and only if their symmetric difference is exactly 2, i.e. $B' = B \setminus \{x\} \cup \{y\}$, for some $x, y \in S$.

Again, the reader is referred to [7] [10] [18] [19] [20] [23] [26], for further information about the bases-exchange graph.

Definition: For a graph $G(V, E)$ and for $X \subseteq V$ define the cutset $C(X)$ of X in the usual way :

$$C(X) = \{(u_1, u_2) \in E : u_1 \in X, u_2 \in V \setminus X\}$$

Definition: The cutset-expansion (or simply expansion) of a graph $G(V, E)$ is :

$$\min_{X \subseteq V, |X| \leq \frac{|V|}{2}} \frac{|C(X)|}{|X|}$$

To the best of our knowledge, there is no counter-example to the following conjecture that was proposed in [18] :

Conjecture: (Matroid-Expansion Conjecture) For any matroid $\mathcal{M}(S, \mathcal{B})$, the cutset expansion of the bases-exchange graph $G(\mathcal{M})$ is at least 1.

The above conjecture has been shown to hold for partition matroids and their truncations [18], and in slightly weaker form for graphic matroids [2] [4].

Clearly, a positive resolution of the conjecture, even in slightly weaker form or for special classes of matroids (vectorial, transversals, graphic and their truncations etc.) would be of major combinatorial interest. Furthermore, it was pointed out in [18] that it would also be of fundamental algorithmic significance:

Fact: [18] If for all matroids $\mathcal{M}(S, \mathcal{B})$ the bases-exchange graph $G(\mathcal{M})$ possesses cutset-expansion inverse polynomial in $|S|$, then for any matroid $\mathcal{M}(S, \mathcal{B})$ there exists an efficient algorithm to approximate $|\mathcal{B}|$, given an independence oracle for \mathcal{M} .

The above assertion follows by standard techniques on random walks on expanders [1] [16] [21], and the well known equivalence of uniform generation and approximate counting for self-reducible combinatorial structures [3] [12]. The reader is referred to [18] for further explanations.

In particular, several unsolved counting problems (including network reliability which is known to be *NP*-hard [24]) can be reduced to counting the number of bases of suitably chosen matroids. So, if the Matroid-Expansion Conjecture holds, then all such problems can be approximated by efficient Monte-Carlo algorithms.

Our work was precisely motivated by the question of resolving the Matroid-Expansion Conjecture.

3 The Ratios-Criterion

Expansion properties of various families of graphs have always been hard to establish [5] [6] [8] [11] [13] [14] [15] [18]. The apparent hardness is due to the fact that expansion is a global property, while the adjacencies of the graph in question are usually defined in a local manner (especially for graphs with intricate combinatorial structure like the bases-exchange graph). Therefore, an expansion argument typically aims to infer global statements by isolating known relevant and usually local structure. It was in this sense that the geometric features of the bases-exchange graph were isolated in [18] (and in fact, it was partial geometric evidence that led to the matroid-expansion conjecture). While geometry has still not resolved the conjecture, we have isolated here yet another criterion for matroid-expansion, the Ratios-Criterion: “If for all $A : A \subseteq \mathcal{B}_e$, $|\Gamma_e(A)|/|A| \geq |\mathcal{B}_e|/|\mathcal{B}_e|$, then $G_e(\mathcal{M})$ is a cut-set expander”. We believe that this criterion seems more local and suitable for matroids for the following reasons:

(i) It is more local because it is the core of an inductive argument by imposing structure to a reasoning about exchanges that involve a particular element e .

(ii) It is more suitable for matroids because it seems to reduce (in a way particularly convenient for matroids) the regularity that needs to be exhibited in an expansion argument. More specifically, as mentioned on the introduction, matroids satisfy certain non-trivial properties for single-step exchanges that are not known to hold for sequences of exchanges (the reader is referred to [25] for a convincing collection of such properties). For example, among others, matroids are known to possess “single-step complementary properties”: For every pair of bases B and B' and for every $x \in B$, there exists $y \in B'$ such that both $B \setminus \{x\} \cup \{y\}$ and $B' \setminus \{y\} \cup \{x\}$ are bases ([25], Chapter 1). Such a property, coupled with the restriction to exchanges involving a specific element e as required by the Ratios-Criterion, can be potentially used to a single-step analogue of a Jerrum and Sinclair type of expansion argument. Along the Jerrum and Sinclair style of reasoning, expansion of a set A is established by exhibiting a substantial number of edge-disjoint paths from A to its complement [5] [11] [18]. In such a setup for the bases-exchange graph $G(\mathcal{M})$, it would be required to reason about the regularity of “complementary” properties holding all along paths that are appropriately defined between any two bases B and B' . In contrast, the Ratios-Criterion requires to reason only about the regularity of the first step of such paths, for which at least complementary properties are in fact known to hold.

We proceed with the technical statements :

Definition: Say that a matroid $\mathcal{M}(S, \mathcal{B})$ enforces ratios if for all $e \in S$ and for all $A \subseteq \mathcal{B}_e$, A enforces ratios with respect to e , i.e.,

$$\frac{|\Gamma_e(A)|}{|A|} \geq \frac{|\mathcal{B}_e|}{|\mathcal{B}_e|} \quad (1)$$

Lemma: *If all matroids enforce ratios, then the dual of (1) is also always true : For all matroids $\mathcal{M}(S, \mathcal{B})$, for all $e \in S$, and for any $A \subseteq \mathcal{B}_e$*

$$\frac{|\Gamma_e(A)|}{|A|} \geq \frac{|\mathcal{B}_e|}{|\mathcal{B}_e|} \quad (2)$$

where $\Gamma_e(A)$ is the set of vertices in \mathcal{B}_e that are connected with some vertex in A by an edge of $G(\mathcal{M})$.

PROOF. (2) is obtained by applying condition (1) to the dual $\mathcal{M}^*(S, \mathcal{B}^*)$ of $\mathcal{M}(S, \mathcal{B})$, where $B \in \mathcal{B}^*$ if and only if $S \setminus B \in \mathcal{B}$, and by noticing that $\mathcal{M}(S, \mathcal{B})$ is isomorphic to $\mathcal{M}^*(S, \mathcal{B}^*)$. \square

Theorem: (The Ratios-Criterion) : *If all matroids enforce ratios, then all matroids have cutset-expansion at least 1.*

PROOF. Consider a matroid $\mathcal{M}(S, \mathcal{B})$ and let $X \subseteq \mathcal{B}$ be such that

$$|X| \leq \frac{|\mathcal{B}|}{2} \quad (3)$$

We wish to show

$$|C(X)| \geq |X| \quad (4)$$

Let $X_e = X \cap \mathcal{B}_e$ and $X_{\bar{e}} = X \cap \mathcal{B}_{\bar{e}}$. Let $C_e(X)$ denote the set of edges in $G(\mathcal{M})$ that are incident to both X_e and $\mathcal{B}_e \setminus X_e$. Let $C_{\bar{e}}(X)$ denote the set of edges in $G(\mathcal{M})$ that are incident to both $X_{\bar{e}}$ and $\mathcal{B}_{\bar{e}} \setminus X_{\bar{e}}$. Let $C_{e\bar{e}}(X)$ denote the set of edges in $G(\mathcal{M})$ that are incident to both X_e and $\mathcal{B}_{\bar{e}} \setminus X_{\bar{e}}$. Finally let $C_{\bar{e}e}(X)$ denote the set of edges in $G(\mathcal{M})$ that are incident to both $X_{\bar{e}}$ and $\mathcal{B}_e \setminus X_e$. Clearly, the above sets define a partition of $C(X)$. Therefore:

$$|C(X)| = |C_e(X)| + |C_{\bar{e}}(X)| + |C_{e\bar{e}}(X)| + |C_{\bar{e}e}(X)| \quad (5)$$

The proof will bound from below $|C(X)|$ by lower-bounding appropriately chosen terms of the right-hand-side of (5). For this we proceed inductively on the size of the ground-set of $\mathcal{M}(S, \mathcal{B})$. The basis of the induction is trivial, and the hypothesis asserts that all matroids over ground-sets of size strictly smaller than $|S|$ have cutset-expansion at least 1.

Case 1 : $|X_e| \leq \frac{|\mathcal{B}_e|}{2}$, and $|X_{\bar{e}}| \leq \frac{|\mathcal{B}_{\bar{e}}|}{2}$.

The inductive hypothesis provides satisfactory lower bounds for $|C_e(X)|$ and $|C_{\bar{e}}(X)|$, and (5) yields a lower bound on $|C(X)|$. In particular, realize that the matroids $\mathcal{M}(S, \mathcal{B}_e)$ and $\mathcal{M}(S \setminus \{e\}, \mathcal{B}_{\bar{e}})$ are isomorphic to matroids over ground-set $S \setminus \{e\}$. Therefore the inductive hypothesis applies to $\mathcal{M}(S, \mathcal{B}_e)$ for X_e , and $\mathcal{M}(S \setminus \{e\}, \mathcal{B}_{\bar{e}})$ for $X_{\bar{e}}$, thus yielding $|C_e(X)| \geq |X_e|$ and $|C_{\bar{e}}(X)| \geq |X_{\bar{e}}|$. Now (5) suggests that $|C(X)| \geq |X|$, which completes the argument for case 1.

Case 2 : $|X_e| > \lfloor \frac{|\mathcal{B}_e|}{2} \rfloor$, and $|X_{\bar{e}}| < \lceil \frac{|\mathcal{B}_{\bar{e}}|}{2} \rceil$.

By the remarks of the previous case, the inductive hypothesis applies to $\mathcal{M}(S, \mathcal{B}_e)$ for $\mathcal{B}_e \setminus X_e$ and $\mathcal{M}(S \setminus \{e\}, \mathcal{B}_{\bar{e}})$ for $X_{\bar{e}}$, thus yielding :

$$\begin{aligned} |C_e(X)| &\geq |\mathcal{B}_e \setminus X_e| = |\mathcal{B}_e| - |X_e| && \text{and} \\ |C_{\bar{e}}(X)| &\geq |X_{\bar{e}}| \end{aligned} \quad (6)$$

Clearly, the bounds in (6) are not strong enough to establish the desired bound for $|C(X)|$. It is at this point that we will use the enforcement of ratios (conditions (1) and (2)) and obtain an additional lower bound for the cross expansion $|C_{e\bar{e}}(X)|$.

Consider some subset A of X_e $|A| = \lceil \frac{|\mathcal{B}_e|}{2} \rceil$. Condition (1) implies that $|\Gamma_e(A)| \geq |A| |\mathcal{B}_{\bar{e}}| / |\mathcal{B}_e| = \lceil \frac{|\mathcal{B}_e|}{2} \rceil$. Therefore, there exists a subset A_1 of $\Gamma_e(A)$ such that $A_1 \cap X_{\bar{e}} = \emptyset$ and $|A_1| = \lceil \frac{|\mathcal{B}_e|}{2} \rceil - |X_{\bar{e}}|$. Let $C_{e\bar{e}, A_1}(X)$ be the set of edges incident to X_e and A_1 . Clearly,

$$|C_{e\bar{e}, A_1}(X)| \geq |A_1| = \lceil \frac{|\mathcal{B}_e|}{2} \rceil - |X_{\bar{e}}| \quad (7)$$

Let $A' = \mathcal{B}_{\bar{e}} \setminus (A_1 \cup X_{\bar{e}})$. Of course, $|A'| = \lfloor \frac{|\mathcal{B}_{\bar{e}}|}{2} \rfloor$. Condition (2) implies that there exists some subset A'_1 of $\Gamma_e(A')$ such that $A'_1 \subseteq X_e$ and $|A'_1| = |X_e| - \lfloor \frac{|\mathcal{B}_{\bar{e}}|}{2} \rfloor$. Now if $C_{e\bar{e}, A'_1}(X)$ is the set of edges incident to A'_1 and A' , it is clear that

$$|C_{e\bar{e}, A'_1}(X)| \geq |X_e| - \lfloor \frac{|\mathcal{B}_{\bar{e}}|}{2} \rfloor \quad (8)$$

Moreover, notice that $C_{e\bar{e}, A_1}(X)$ and $C_{e\bar{e}, A'_1}(X)$ are disjoint. Therefore (7) and (8) imply:

$$\begin{aligned} |C_{e\bar{e}}(X)| &\geq |C_{e\bar{e}, A_1}(X)| + |C_{e\bar{e}, A'_1}(X)| \\ &\geq \lceil \frac{|\mathcal{B}_e|}{2} \rceil - |X_{\bar{e}}| + |X_e| - \lfloor \frac{|\mathcal{B}_{\bar{e}}|}{2} \rfloor \end{aligned} \quad (9)$$

Finally, (5), (6), and (9) imply :

$$\begin{aligned} |C(X)| &\geq |\mathcal{B}_e| - |X_e| + |X_{\bar{e}}| + \lceil \frac{|\mathcal{B}_{\bar{e}}|}{2} \rceil - |X_{\bar{e}}| + |X_e| - \lfloor \frac{|\mathcal{B}_e|}{2} \rfloor \\ &= \lceil \frac{|\mathcal{B}_e|}{2} \rceil + \lceil \frac{|\mathcal{B}_{\bar{e}}|}{2} \rceil \\ &\geq \frac{|\mathcal{B}|}{2} \\ &\geq |X_e| + |X_{\bar{e}}| \\ &= |X| \end{aligned}$$

This last bound completes the treatment of Case 2.

Case 3 : $|X_e| < \lceil \frac{|\mathcal{B}_e|}{2} \rceil$, and $|X_{\bar{e}}| > \lfloor \frac{|\mathcal{B}_{\bar{e}}|}{2} \rfloor$. This case is identical to Case 2, which completes the proof of Theorem 4.3. \square

Remark : A Ratios-Condition similar to (1) and (2) can be shown to hold for the polytopes of order ideals, independent sets, partition matroids and their truncations, and matchings and their “slices” that were examined in [18]. This results in cutset expansion factors 1 for these polytopes (while in [18] it was shown that these polytopes have “strong cutset expansion” 1, hence, cutset expansion only 1/2). The reasoning is inductive and uses single-step complementary point arguments that are known to hold for the above polytopes. The reader is referred to [17] for the complete proof.

4 Matroid Connectivity and the e -Restricted Bases-Exchange graph

The Ratios-Criterion that was developed in the previous section was based on the exchanges involving only one specific element e of the ground set S . In particular, the graph that was underlying throughout the Ratios-Criterion was the “ e -restricted bases-exchange graph”, formally defined as follows :

Definition: For a matroid $\mathcal{M}(S, \mathcal{B})$ and for some $e \in S$ the e -restricted bases-exchange graph : $G_e(\mathcal{M})$ is a graph on vertex-set $\mathcal{B}(= \mathcal{B}_e \cup \mathcal{B}_{\bar{e}})$, and two bases B and B' are connected by an edge if and only if :

- $B \in \mathcal{B}_e$ and $B' \in \mathcal{B}_{\bar{e}}$, and
- B and B' are connected by an edge in $G(\mathcal{M})$.

In terms of the e -restricted bases-exchange graph the ratios enforcement condition in the Ratios-Criterion states that for all subsets A of \mathcal{B}_e the following holds :

$$\frac{|\Gamma_e(A)|}{|A|} \geq \frac{|\mathcal{B}_{\bar{e}}|}{|\mathcal{B}_e|}$$

It is easy to see that if the bases-exchange graph indeed passes the Ratios-Criterion, then it is necessary that when $X(= X_e \cup X_{\bar{e}})$ is an entire connected component of $G_e(\mathcal{M})$ then this component enforces ratios with equality, i.e., $|X_e|/|X_{\bar{e}}| = |\mathcal{B}_e|/|\mathcal{B}_{\bar{e}}|$. While the general condition remains open, we show here that the above non-trivial necessary condition is in fact true. The key ingredient in our proof is the characterization of the connected components of $G_e(\mathcal{M})$, which, in turn, results in an alternative definition of matroid connectivity.

A very simple (and very instructive) example of a matroid is the *Graphic Matroid* of a graph $G(V, E)$. The ground set of this matroid is the set E of the edges of G and the bases are the spanning trees. Biconnectivity in graphs is a well known property; connectivity in matroids is analogous to biconnectivity in graphs : A matroid $\mathcal{M}(S, \mathcal{B})$ is *connected* if there exists a

circuit through every pair of elements in the ground set. Furthermore, if a matroid is not connected, then its ground set can be partitioned uniquely into connected components (for graphic matroids this corresponds to a partitioning of the edges into biconnected components). The original matroid induces a matroid on each component in a natural way and the original matroid can be reconstructed as the direct sum of the components. (Again it is easy to see that spanning trees of the original graph are simply the union of spanning trees of biconnected components.)

Our main theorem relates the connectivity of the e -restricted bases-exchange graph to the connectivity of the matroid.

Theorem: (Connectivity Theorem) *The e -restricted bases-exchange graph of a matroid \mathcal{M} is connected if and only if the matroid is connected.*

Consequently, if the matroid is not connected then the structure of the connected components of $G_e(\mathcal{M})$ can be characterized as follows :

Theorem: *Let $\mathcal{M}_1(S_1, \mathcal{B}_1)$ be the connected component containing an element e in a matroid \mathcal{M} and let $\mathcal{M}_2(S_2, \mathcal{B}_2)$ be the induced matroid on the remaining elements of the ground-set ($S_2 = S \setminus S_1$). Then $G_e(\mathcal{M})$ contains exactly one connected component for each basis B_2 in \mathcal{B}_2 and each connected component is isomorphic to $G_e(\mathcal{M}_1)$.*

Again, by Theorem 4.3 the following necessary condition for the e -restricted bases exchange graph to pass the Ratios Criterion is true:

Corollary: (Necessary Condition) *Every connected component of $G_e(\mathcal{M})$ enforces ratios with equality, i.e. if $X = X_e \cup X_{\bar{e}}$ is a connected component of $G_e(\mathcal{M})$, then $|X_e|/|X_{\bar{e}}| = |\mathcal{B}_e|/|\mathcal{B}_{\bar{e}}|$.*

In the rest of the section we give the technical details of the proof of Theorems 4.2, 4.3 and Corollary 4.4.

4.1 Preliminaries

Recall that a subset of the ground-set is *independent* if it is the subset of a base. A set that is not independent is called *dependent*.

Definition: *A circuit in a matroid \mathcal{M} is a minimal dependent subset of S .*

In Definition 2.1 a matroid was introduced in terms of its bases. Equivalently a matroid can be introduced in terms of its circuits (e.g. see [25]):

Definition: (Circuit Axioms) *A collection \mathcal{C} of subsets of S is the set of circuits of a matroid on S if and only if for all $X, Y \in \mathcal{C}$*

- (i) X is not a subset of Y
- (ii) *If $z \in X \cap Y$ and $y \in X \Delta Y$, then there exists a circuit $Z \in \mathcal{C}$ such that $Z \subset X \cup Y \setminus \{z\}$ and $y \in Z$ (here Δ denotes symmetric difference).*

The circuits of a matroid define a natural relation R on the ground-set. The equivalence classes of this relation are the connected components of the matroid.

Definition: Let $R \subset S \times S$ be such that $e_1 R e_2$ if and only if $\exists C \in \mathcal{C}$ such that $e_1, e_2 \in C$.

It is well known that R defined on $S \times S$, as above, is an equivalence relation.

Definition: The equivalence classes of the relation R induce a partition of S whose partition classes are the connected components of S .

Definition: A matroid $\mathcal{M}(S, \mathcal{B})$ is said to be connected if and only if S has exactly one connected component.

In the following subsections we show the sufficiency and necessity of matroid connectivity for the connectedness of $G_e(\mathcal{M})$. In particular we will establish Theorem 4.2 which follows from Lemma 4.17 and Lemma 4.20.

We first show sufficiency, i.e. that a connected matroid results in a connected e -restricted bases-exchange graph.

4.2 Sufficiency

Let $\mathcal{M}(S, \mathcal{B})$ be a connected matroid. We show that every pair of bases that are adjacent in $G(\mathcal{M})$ are connected by a path of length $O(|S|)$ in $G_e(\mathcal{M})$. Since the bases-exchange graph is connected, this implies that the e -restricted bases-exchange graph is also connected.

In the proof we shall first exhibit the existence of “chains” of circuits in \mathcal{M} that link adjacent bases of $G(\mathcal{M})$. We shall further show that these chains are sufficient to establish paths in $G_e(\mathcal{M})$ between adjacent bases of $G(\mathcal{M})$.

Definition: The *rank function* of a matroid is a function $\rho : 2^S \rightarrow \mathbb{N}_0$ such that for all $X \subset S$:

$$\rho(X) = \max(|Y| : Y \subset X, Y \in \mathcal{I})$$

In words, the rank of a set is the size of the maximal independent set that is contained in the set.

Definition: A sequence of circuits C_1, C_2, \dots, C_k is a *chain of length k* if $\rho(C_i \cup C_{i+1}) = |C_i \cup C_{i+1}| - 2$ and $C_i \cap C_{i+1} \neq \emptyset$, for all $i : 1 \leq i \leq k$.

In the case of graphic matroids a chain is a sequence of cycles where intersection of successive cycles is a path in the graph.

Fact: The conditions $\rho(C_i \cup C_{i+1}) = |C_i \cup C_{i+1}| - 2$ and $C_i \cap C_{i+1} \neq \emptyset$ guarantee that $C_i \Delta C_{i+1}$ is contained in a circuit.

PROOF. Let $z \in C_i \cap C_{i+1}$. Then the circuit axioms guarantee that for every element $x \in C_i \Delta C_{i+1}$ there exists a circuit contained in $C_i \cup C_{i+1} \setminus \{z\}$ that contains x . But the condition $\rho(C_i \cup C_{i+1}) = |C_i \cup C_{i+1}| - 2$, implies that there is a unique circuit in $C_i \cup C_{i+1} \setminus \{z\}$, which implies that $C_i \Delta C_{i+1}$ is contained in a circuit. \square

Recall that for a basis $B \in \mathcal{B}$ and an element $x \in S \setminus B$ there is a unique circuit in $B \cup \{x\}$.

Definition: For a basis B and an element $x \in S \setminus B$, the unique circuit in $B \cup \{x\}$ is called the fundamental circuit of (B, x) . This will be henceforth denoted by $C(B, x)$.

Definition: Let $e \in S$ and let B and B' be two bases in \mathcal{M} such that $B \Delta B' = \{e_1, e_2\}$, where $e_1 \in B$ and $e_2 \in B'$. Define B and B' to be linked with respect to e by a chain of length k , if there exists a sequence of elements $x_1, x_2, \dots, x_k = e_2$, such that $C(B, x_1), C(B, x_2), \dots, C(B, x_k)$ is a chain of length k and $e \in C(B, x_1)$. We shall also say that the chain $C(B, x_1), C(B, x_2), \dots, C(B, x_k)$ links e to e_1 with respect to the basis B .

Claim: Let B, B' be two bases in \mathcal{M} such that $B \Delta B' = \{e_1, e_2\}$, where $e_1 \in B$ and $e_2 \in B'$. Then there exists a chain of length at most $|S|$ which links B and B' .

PROOF. Follows from Lemma 6.2 in [?]. □

Claim: If B, B' are linked by a chain of length l , then there exists a path of length at most $4l$ between B and B' in $G_e(\mathcal{M})$.

PROOF. We use induction on l . The case $l = 1$ is trivial.

Consider B and B' that are linked by a chain of length l .

Let the sequence of edges forming the chain be x_1, x_2, \dots, x_l and let the chain be $C_1 = C(B, x_1), C_2 = C(B, x_2), \dots, C_l = C(B, x_l)$. Assume further that if $|i - j| > 1$ then $C_i \cap C_j = \emptyset$ (otherwise there exists a smaller chain linking B and B').

Consider $y \in C_1 \cap C_2$ (notice $y \in B$). Now consider the bases $B1 = B \cup \{x_1\} \setminus \{y\}$ and $B1' = B' \cup \{x_1\} \setminus \{y\}$. $B1$ (resp. $B1'$) can be reached from $B1$ (resp. $B1'$) by exchanges involving e . We show below that B and B' are linked by a chain of length $l - 1$.

Consider the sequence of edges x_2, x_3, \dots, x_l , and the circuits $C1_1 = C(B1, x_2), C1_2 = C(B1, x_3), \dots, C1_{l-1} = C(B1, x_l)$. Now notice:

- $C1_i = C_{i+1}, \forall i > 1$: This follows from the fact that B is different from $B1$ only in $\{x_1, y\}$.
- $C1_1 \cap C1_2 \neq \emptyset$: This is true since $C1_1 \Delta C2 \subset C1_1$ and $C3 \cap C2 \cap C1 = \emptyset$.

Thus $C1_1, C1_2, \dots, C1_{l-1}$ is a chain of length $l - 1$ that links $B1$ with $B1'$. This completes our proof. □

All the above imply

Lemma: The e -restricted bases-exchange graph of a matroid \mathcal{M} is connected if \mathcal{M} is connected.

PROOF. Follows from Claims 4.15 and 4.16. □

Next we show that $G_e(\mathcal{M})$ is not connected if \mathcal{M} is not connected.

4.3 Necessity

Let $\mathcal{M}(S, \mathcal{B})$ be a disconnected matroid and let S' be a minimal non-empty separator of \mathcal{M} not containing e . Then, for any base B , no exchange involving e will change $B \cap S'$. Thus if there exist two bases B_1 and B_2 such that $B_1 \cap S' \neq B_2 \cap S'$, then there is no path connecting them. This proves that the connectedness of the matroid is a necessary condition for the connectedness of the e -restricted basis-exchange graph.

Theorem 4.3 *Let $\mathcal{M}_1(S_1, \mathcal{B}_1)$ be the connected component containing an element e in a matroid \mathcal{M} and let $\mathcal{M}_2(S_2, \mathcal{B}_2)$ be the induced matroid on the remaining elements of the matroid ($S_2 = S \setminus S_1$). Then $G_e(\mathcal{M})$ contains exactly one connected component for each basis B_2 in \mathcal{B}_2 and each connected component is isomorphic to $G_e(\mathcal{M}_1)$.*

PROOF. The proof follows from the fact that all bases of \mathcal{M} are formed by the union of bases of \mathcal{M}_1 and \mathcal{M}_2 . Thus for a given basis B_2 of \mathcal{M}_2 , all the vertices of $G_e(\mathcal{M})$ that are of the form $\{B_1 \cup B_2 : B_1 \text{ is a basis of } \mathcal{M}_1\}$ form a connected component of $G_e(\mathcal{M})$. The edges in $G_e(\mathcal{M})$ are all based on circuits of \mathcal{M}_1 . Thus every such basis B_2 in \mathcal{B}_2 induces a component in $G_e(\mathcal{M})$. \square

4.4 Connected Components of $G_e(\mathcal{M})$

We are now in a position to establish the necessary condition :

Corollary 4.4 (Necessary Condition) Each connected component of $G_e(\mathcal{M})$ enforces ratios with equality i.e. if $X = X_e \cup X_{\bar{e}}$ are the vertices of a connected component of $G_e(\mathcal{M})$, then $|X_e|/|X_{\bar{e}}| = |\mathcal{B}_e|/|\mathcal{B}_{\bar{e}}|$.

PROOF. Follows from the fact that all components are isomorphic and hence the ratio of vertices in either partition is the same in all components. \square

5 Conclusion

We investigate the connectivity property of matroids and their bases-exchange graph and obtain an alternative definition for matroid connectivity. The motivation for our work was to resolve the matroid-expansion conjecture of [18]. While the conjecture still remains open, we provide a new local expansion criterion that appears suitable for matroids. Resolving the matroid expansion conjecture is of fundamental algorithmic significance. —————

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